On 0-rotatable caterpillars with diameter at least 7

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Abstract

A graceful labelling of a tree T is an injective function $f\colon V(T) \to \{0, \ldots, |E(T)|\}$ such that $\{|f(u) - f(v)|\colon uv \in E(T)\} = \{1, \ldots, |E(T)|\}$. A tree T is said to be 0-rotatable if, for each $v \in V(T)$, there exists a graceful labelling f of T such that $f(v) = 0$. In this work, it is proved that if T is a caterpillar with $diam(T) \geq 7$ and, for every non-leaf vertex $v \in V(T)$, the number of leaves adjacent to v is at least $2 + 2((diam(T) - 1) \bmod 2)$, then T is 0-rotatable. This result reinforces the conjecture that every caterpillar with diameter at least five is 0-rotatable.

1 Introdução

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If $u, v \in V(G)$ are the ends of $e \in E(G)$, we also denote e by uv or vu. A *labelling* of G is an injective function $f\colon V(G) \to \mathbb{Z}_{\geq 0}$. Under labelling f, the *label* of a vertex $v \in V(G)$ is $f(v)$, and the *(induced) label* of an edge $uv \in E(G)$ is the absolute difference of the labels of its ends, $|f(u) - f(v)|$. Given a labelling f of G, denote by L_V^f the set of vertex labels under f and denote by L_E^f the set of induced edge labels under f. Labelling f is *graceful* if $L_V^f \subseteq \{0, \ldots, |E(G)|\}$ and $L_E^f = \{1, \ldots, |E(G)|\}$. A labelling f of G is an α*-labelling* if f is graceful and there exists an integer $k \in \{0, \ldots, |E(G)|\}$ such that, for each edge $uv \in E(G)$, $f(u) \leq k < f(v)$.

Graceful labellings and α-labellings were introduced by Rosa [8] in 1967. In his seminal paper, Rosa posed the famous *Graceful Tree Conjecture*, which states that all trees are *graceful*, that means all trees have a graceful labelling. The author proved that the Graceful Tree Conjecture is a strengthened version of the well-known *Ringel-Kotzig Conjecture*, which states that K_{2m+1} has a cyclic decomposition into subgraphs isomorphic to a given tree T with m edges. The Graceful Tree Conjecture is a very important open problem in Graph Theory, with hundreds of papers about it [3].

It is well-known the importance of label 0 on graceful labellings: for example, it is easy to grow a gracefully labelled tree T by adding k new leaves with labels $|E(T)| + 1, \ldots, |E(T)| + k$ to the 0-labelled vertex; also, it is possible to combine any tree with an α-labelling and any tree with a graceful labelling, by identifying the vertices labelled 0, such that the resultant tree is graceful [5]. Considering the importance of label 0, we say that a tree T is 0*-rotatable* if, for each $v \in V(T)$, there exists a graceful labelling f of T such that $f(v) = 0$.

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The 0-rotatability of trees was first investigated by Rosa [8] and it is still an open problem even for caterpillars. In 1977, the author proved that all paths are 0-rotatable [9]. Posteriorly, Chung and Hwang [2] showed that every caterpillar whose non-leaf vertices have the same degree is 0-rotatable. In 2004, Bussel [1] showed that all trees with diameter at most three are 0-rotatable and, additionally, showed that there exist non-0-rotatable trees with diameter four. In fact, Bussel showed that all non-0-rotatable trees with at most 14 vertices either are caterpillars with diameter four, or are trees formed by identifying the central vertex of a non-0-rotatable tree of diameter four with the end of a path P_n, $n \geq 1$. Based on these results, the author conjectured that all non-0-rotatable trees belong to these two families of trees. Note that if Bussel's conjecture is true, then it implies that every caterpillar with diameter at least five is 0-rotatable.

In a previous work [6], we investigated Bussel's Conjecture restricted to caterpillars and proved that all caterpillars with diameter five or six are 0-rotatable. In this work, this investigation is taken further and it is proved that if T is a caterpillar with $diam(T) \geq 7$ and, for every non-leaf vertex $v \in V(T)$, the number of leaves adjacent to v is at least $2 + 2((diam(T) - 1) \bmod 2)$, then T is 0-rotatable. This result reinforces the conjecture that every caterpillar with diameter at least five is 0-rotatable. In particular, this family shows that, for each integer $d \geq 7$, there exist 0-rotatable caterpillars with diameter d and arbitrary number of vertices.

In the next section, we present some additional definitions as well as classic results and techniques that are used in our proofs. The main results are presented in Section 3.

2 Preliminaries

Let T be a tree with graceful labelling f. By the definition, f is injective and, since T is a tree, f is also onto. Therefore, f^{-1} is well-defined and it is used to refer elements of $V(G)$.

The *complementary labelling* of f is the labelling $\overline{f}$ defined by $\overline{f}(v) = |E(T)| - f(v)$ for each $v \in V(T)$. Note that the complementary labelling is also a graceful labelling since: (i) $\overline{f}(v)$ is an injection from $V(T)$ to $\{0, \ldots, |E(T)|\}$; and (ii) for each $uv \in E(T)$, $|\overline{f}(u) - \overline{f}(v)| = |(|E(T)| - f(u)) - (|E(T)| - f(v))| = |f(v) - f(u)|$.

Let $v \in V(T)$. Denote by $N^k_T(v)$ the set of neighbours of v with degree k. The *distance* $d(u, v)$ between two vertices $u, v \in V(T)$ is the number of edges in the unique path connecting u and v in T. The *diameter* of T is defined as $diam(T) = \max\{d(u, v)\colon u, v \in V(T)\}$. The *center* of T is the set of all vertices $u \in V(T)$ where the greatest distance $d(u, v)$ to other vertices $v \in V(T)$ is minimal. We say that $u \in V(T)$ is a *central vertex* of T if u belongs to the center of T. The *base* of T, B_T, is the tree obtained from T by removing all of its leaves. A *path*, P_n, is a tree whose vertices can be arranged in a linear sequence such that two vertices in P_n are adjacent if and only if they are consecutive in the sequence. We say that a tree T is a *caterpillar* if its base is isomorphic to a path. The next results consider graceful and α-labellings of paths and caterpillars.

Lemma 1 (Rosa [9]). *Let P_n be a path, $n \geq 1$, and let $v \in V(P_n)$. Then,*

(i) there exists an α-labelling f of P_n such that $f(v) = 0$ if and only if v is not the central vertex of P_5.

(ii) if v is the central vertex of P_5, then P_5 has a graceful labelling f such that $f(v) = 0$. □

Lemma 2. *Let T be a caterpillar such that $N^1_T(v) \geq 1$ for each $v \in V(B_T)$. Let $S \subseteq N^1_T(v)$ and $T_v = T \backslash S$. If, for every $v \in V(B_T)$, T_v has a graceful labelling f with $f(v) = 0$, then T is 0-rotatable.* □

The main technique used in our proofs is the method of transfers, defined as follows. Let u, v, w be distinct vertices of a tree T, such that w is adjacent to u and is not adjacent to v. We call *transfer* the operation of deleting edge wu from T and adding edge wv. After the transfer operation, we say that vertex w has been *transferred* or *moved* from u to v. For any two distinct vertices u and v of a gracefully labelled tree T, the notation $u \to v$ means that we moved some vertices incident with vertex u to vertex v. If T is graceful, we say that a transfer $u \to v$ is *safe* if the resulting tree is also graceful. The following lemma establishes conditions to perform safe transfers.

Lemma 3 (Hrnčiar and Haviar [4]). *Let f be a graceful labelling of a tree T and let $u, v \in V(T)$ be two distinct vertices. If u is adjacent to (not necessarily distinct) vertices $u_1, u_2 \in V(T)$, such that $u_1 \neq v$, $u_2 \neq v$ and $f(u_1) + f(u_2) = f(u) + f(v)$, then tree T', obtained from T by moving u_1, u_2 from u to v, is also graceful.* □

A $u \to v$ transfer is said to be *of the first type* if the labels of the transferred vertices constitute a set of consecutive integers $k \dots, k+p$ such that $f(u) + f(v) = k + (k+p)$. On the other hand, $u \to v$ is *of the second type* if the labels of the transferred vertices form two consecutive sequences $k, \dots, k+p$ and $l, \dots, l+p$ such that $f(u) + f(v) = k + l + p$. Figure 1 illustrates these concepts.

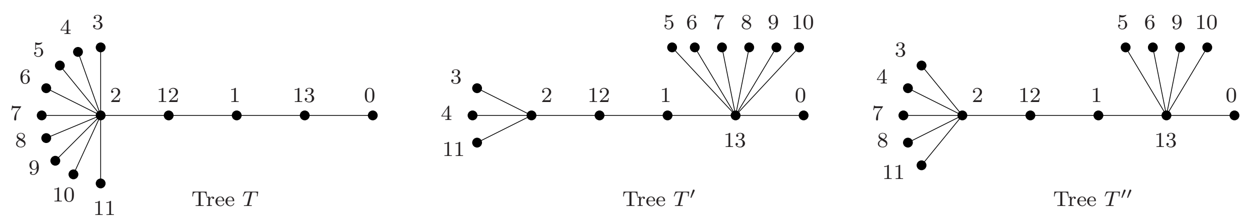

Figure 1: Given tree T with graceful labelling f, tree T' is obtained from T by applying a transfer $f^{-1}(2) \to f^{-1}(13)$ of the first type; and tree T'' is obtained from T by applying a transfer $f^{-1}(2) \to f^{-1}(13)$ of the second type.

The next lemma establishes additional conditions under which it is possible to make safe transfers in a graceful tree.

Lemma 4 (Mishra and Panigrahi [7]). *Let T be a tree with a graceful labelling f satisfying the following two properties:*

(i) there exist distinct vertices in T with labels $a - r_1, \dots, a, b, \dots, b + r_2$ such that $a < b$ and $r_1, r_2 \in \mathbb{Z}_{\geq 0}$;

(ii) the vertex with label a is adjacent to a set of vertices $\mathcal{S}$ with labels $s, \dots, s+p$, such that:

- *(a) $p \geq 2$;*
- *(b) $\{s, \dots, s+p\} \cap \{a - r_1, \dots, a, b, \dots, b + r_2\} = \emptyset$; and*
- *(c) for $0 \leq i \leq \lfloor \frac{p-1}{2} \rfloor$, either $(s+i+1) + (s+p-i) = a+b$ or $(s+i) + (s+p-1-i) = a+b$.*

If $|\mathcal{S}|$ is even, then it is possible to make a sequence of safe transfers of the second type $f^{-1}(a) \to f^{-1}(b) \to f^{-1}(a-1) \to f^{-1}(b+1) \to f^{-1}(a-2) \to f^{-1}(b+2) \to \dots \to f^{-1}(z)$, where $z = a - r_1$ or $z = b + r_2$, keeping a positive even number of vertices of $\mathcal{S}$ at each vertex of the sequence. □

3 Main result

In this section, we prove our main result.

Theorem 5. *If T is a caterpillar with $diam(T) \geq 7$ and $N_T^1(v) \geq 2 + 2((diam(T) - 1) \bmod 2)$ for every $v \in V(B_T)$, then T is 0-rotatable.*

Proof. Let T be a caterpillar as described in the hypothesis. Let $v \in V(B_T)$ and $\{A, B\}$ be a bipartition of B_T such that $|A| \geq |B|$. Let $w_1, w_2 \in N_T^1(v)$ and $T_1 \subset T$ be the tree induced by vertex set $V(T) \backslash L_v$, where

$$L_v = \begin{cases} N_T^1(v), & \text{if } |A| = |B| \text{ or } (|A| \neq |B| \text{ and } v \in A); \\ N_T^1(v) \backslash \{w_1, w_2\}, & \text{otherwise.} \end{cases}$$

By Lemma 2, it suffices to show that T_1 has a graceful labelling f such that $f(v) = 0$. First, note that $B_T = B_{T_1}$. Let $q = |V(B_T)|$. Since $q \geq 6$, by Lemma 1, P has an α-labelling $g\colon V(P) \to \{0, \ldots, q-1\}$ such that $g(v) = 0$. Given the bipartition $\{A, B\}$, adjust notation so that $v \in A$. By the definition of α-labelling and since $g(v) = 0$, all the labels of the vertices in A are smaller than the labels of the vertices in B. In fact, since P is a tree, $L_A^g = \{0, \ldots, q_A - 1\}$ and $L_B^g = \{q_A, \ldots, q_A + q_B - 1\}$, where $q_A = |A|$, $q_B = |B|$, and $q = q_A + q_B$.

Let l be the number of leaves of T_1. Using bipartition $\{A, B\}$, we modify g in order to obtain another labelling h of P as follows:

$$h(u) = \begin{cases} g(u), & \text{if } u \in A; \\ g(u) + l, & \text{if } u \in B. \end{cases}$$

Therefore, $h\colon V(P) \to L_A^h \cup L_B^h$, with $L_A^h = \{0, \ldots, q_A - 1\}$ and $L_B^h = \{q_A + l, \ldots, q - 1 + l\}$. Note that the vertex labels $q_A, \ldots, q_A + l - 1$ are missing in $L_A^h \cup L_B^h$. Moreover, since each vertex in B had its label increased by l, $L_{E(P)}^h = \{1 + l, \ldots, q - 1 + l\}$. Therefore, the induced edge labels $1, \ldots, l$ are missing in $L_{E(P)}^h$.

Let T_2 be the tree obtained by adding l leaves with labels $q_A, \ldots, q_A + l - 1$ to the vertex $h^{-1}(q_A + l)$. Let f be this new labelling. Note that f is a graceful labelling of T_2 since: (i) the labels of the vertices of T_2 are $0, \ldots, q - 1 + l$; (ii) the labels of the l new edges are $1, \ldots, l$; and (iii) the remaining edge labels are $l + 1, \ldots, q - 1 + l$.

Let $u_1 = f^{-1}(q_A + l)$ and $u_2 = f^{-1}(q_A - 1)$. By construction, vertex u_1 is adjacent to a set of leaves with labels $q_A, \ldots, q_A + l - 1$. Since $(q_A + i) + (q_A + l - 1 - i) = f(u_1) + f(u_2)$, by Lemma 3, it is possible to make a safe transfer $u_1 \to u_2$ of the first type. Let $|N_{T_1}^1(u_1)| = 2j$, $j \in \mathbb{Z}_{>0}$. Thus, we move $\frac{l}{2} - j$ pairs of leaves with labels $q_A + j + i$ and $q_A + l - 1 - j - i$ from u_1 to u_2, for $0 \leq i \leq \lfloor \frac{l-2j-1}{2} \rfloor$. Note that the transferred vertices have consecutive integers $q_A + j, \ldots, q_A + l - 1 - j$ as labels. Let T_3 be the tree obtained after this transfer.

Now, considering T_3 and f, we apply Lemma 4 to make a sequence of safe transfers in T_3 so as to obtain a gracefully labelled caterpillar isomorphic to T_1. In order to do this, take $a = q_A - 1$, $b = q_A + l + 1$, $s = q_A + j$, $p = l - 2j - 1$, $r_2 = q_B - 2$,

$$r_1 = \begin{cases} q_A - 2, & \text{if } q \text{ is even or } (q \text{ is odd and } q_A > q_B); \\ q_A - 1, & \text{if } q \text{ is odd and } q_A < q_B; \end{cases}$$

and

$$z = \begin{cases} b + r_2, & \text{if } q \text{ is even or } (q \text{ is odd and } q_A < q_B); \\ a - r_1, & \text{if } q \text{ is odd and } q_A > q_B. \end{cases}$$

By Lemma 4, it is possible to make the sequence of safe transfers $f^{-1}(q_A-1) \to f^{-1}(q_A+l+1) \to f^{-1}(q_A-2) \to f^{-1}(q_A+l+2) \to \ldots \to f^{-1}(z)$ of the second type such that each vertex u of the sequence receives exactly $N^1_{T_1}(u)$ leaves. Remembering that $N^1_{T_1}(u_1) = N^1_{T_3}(u_1)$, we conclude that, after this sequence of transfers, each vertex $u \in V(B_T)$ is adjacent to exactly $N^1_{T_1}(u)$ leaves. Therefore, the resultant graceful tree is isomorphic to T_1 and its graceful labelling f assigns label 0 to vertex v, as required. □

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