On α-labellings of lobsters and trees with a perfect matching

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Abstract

A graceful labelling of a tree T is an injective function $f\colon V(T) \to \{0,\dots,|E(T)|\}$ such that $\{|f(u)-f(v)|\colon uv \in E(T)\} = \{1,\dots,|E(T)|\}$. An α-labelling of a tree T is a graceful labelling f with the additional property that there exists an integer $k \in \{0,\dots,|E(T)|\}$ such that, for each edge $uv \in E(T)$, either $f(u) \le k < f(v)$ or $f(v) \le k < f(u)$. In this work, we prove that the following families of trees with maximum degree three have α-labellings: lobsters with maximum degree three, without Y-legs and with at most one forbidden ending; trees T with a perfect matching M such that the contraction T/M has a balanced bipartition and an α-labelling; and trees with a perfect matching such that their contree is a caterpillar with a balanced bipartition. These results reinforce the conjecture that every tree with maximum degree three and a perfect matching has an α-labelling.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote an edge $e \in E(G)$ by uv where $u,v \in V(G)$ are its endpoints. As usual, the degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. We say that $v \in V(G)$ is a *degree* 3 *vertex* if $d_G(v) = 3$.

A *labelling* of G is an injective function $f\colon V(G) \to \mathbb{Z}_{\ge 0}$. Under labelling f, the *label* of a vertex $v \in V(G)$ is $f(v)$, and the *(induced) label* of an edge $uv \in E(G)$ is the absolute difference of the labels of its endpoints, $|f(u)-f(v)|$. Given a labelling f of G, denote by $\mathcal{L}_V^f$ the set of vertex labels under f and denote by $\mathcal{L}_E^f$ the set of induced edge labels under f. Labelling f is *graceful* if $\mathcal{L}_V^f \subseteq \{0,\dots,|E(G)|\}$ and $\mathcal{L}_E^f = \{1,\dots,|E(G)|\}$. A labelling f of G is an *α-labelling* if f is graceful and there exists an integer $k \in \{0,\dots,|E(G)|\}$ such that, for each edge $uv \in E(G)$, either $f(u) \le k < f(v)$ or $f(v) \le k < f(u)$.

Graceful labellings and α-labellings were introduced by Rosa in 1967 [8]. In his seminal article, Rosa posed the *Graceful Tree Conjecture* which states that all trees are *graceful* (that is, have graceful labellings). The author proved that the Graceful Tree Conjecture is a strenghtened version of the well-known *Ringel-Kotzig Conjecture*: the complete graph K_{2m+1} has a cyclic decomposition into subgraphs isomorphic to a given tree T with m edges. Rosa also proved that, for any positive integer p, if a graph G with m edges has an α-labelling, then there exists a cyclic decomposition of the complete graph K_{2pm+1} into subgraphs isomorphic to G. These results stress the importance of graceful and α-labellings in the study of cyclic decompositions of complete graphs.

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A result that follows directly from the definition of α-labelling is that if G has an α-labelling, then G is bipartite. Therefore, the only graphs which are expected to have such a labelling are bipartite graphs, like trees. However, it is important to remark that there exist graceful trees that do not have α-labellings. The smallest example is the tree obtained by subdividing every edge of the star $K_{1,3}$. It is still not known which trees with maximum degree 3 have α-labellings. Some results in this direction are known [1, 4, 2, 3]. In particular, Brankovic et al. [2] showed that all trees with at most 28 vertices, maximum degree three and a perfect matching have an α-labelling and, based on this result, they posed Conjecture 1.

Conjecture 1 (Brankovic et al. [2]). *All trees with maximum degree three and a perfect matching have an α-labelling.*

Brinkman et al. [4] prove that if we take a tree with $4k$ vertices, all with odd degree, and subdivide each of its edges exactly once, then the resulting tree has no α-labelling. Letting $\mathcal{F}$ be the set of such trees T, they further prove that, if T is any tree such that $15 \le |V(T)| \le 36$ and T has maximum degree three, then either T has an α-labelling or $T \in \mathcal{F}$. They posed the following question.

Question 2 (Brinkmann et al. [4]). *Do all trees with at least 15 vertices, maximum degree three, and without α-labellings belong to family $\mathcal{F}$?*

A spine in a tree T is a longest path in T. A tree T is *k-distant* if there is a spine P such that every vertex of T is distance at most k from a vertex in P. (It is a simple exercise to show that every spine will work.) The *paths* are precisely 0-distant trees and 1-distant trees are exactly the *caterpillars*. The 2-distant trees are *lobsters* and a principal part of this work is the study of α-labellings in lobsters with maximum degree 3. It is known that all paths and caterpillars have α-labellings but the same is not true for lobsters [8]. In particular, Huang et al. [6] showed that all lobsters of diameter four that are not caterpillars do not have an α-labelling.

In order to state our results on α-labellings of lobsters, some additional definitions are needed. Let G be a lobster with $\Delta(G) = 3$. The *legs* of G are the non-trivial connected components obtained by removing the edges of its spine. Note that, since $\Delta(G) = 3$, the legs of G are isomorphic to a path with two vertices, a path with three vertices, or the bipartite graph $K_{1,3}$, and are called *1-leg*, *2-leg* and *Y-leg*, respectively. If the spine P is the path $(v_1, v_2, \ldots, v_t)$, then each leg contains exactly one of $v_2, v_3, \ldots, v_{t-1}$ as a vertex of degree 1 in the leg. An *ending* of L consist of a subpath P' of P containing either v_1 or v_t, together with all the legs having a vertex in P'. There are six *forbidden endings*: two have $P' = (v_1, v_2, v_3)$ with a 2-leg containing v_3 and v_2 is either not in any leg or it is in a 1-leg; and one has $P' = (v_1, v_2, v_3, v_4)$, with v_4 in a 2-leg and both v_2 and v_3 in 1-legs. The other three are the reflections of these containing v_t. See Figure 1 for diagrams of these last three.

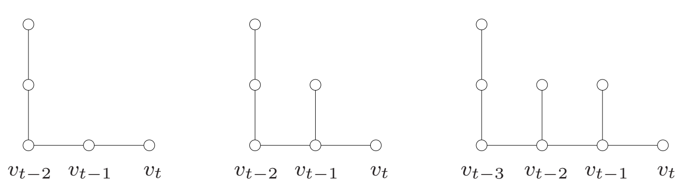


Figure 1: Three forbidden endings.

In this work, we prove the following result on α-labellings of lobsters with maximum degree three:

Theorem 3. *Let G be a lobster with $\Delta(G) = 3$ and without Y-legs. If G has at most one forbidden ending, then G has an α-labelling.*

A tree T is *balanced* if its bipartition $\{A, B\}$ has the property that $|A|$ and $|B|$ differ by at most one. The *contree* of T is the tree obtained from T by contracting the edges of M. In this work, we also prove the following results on α-labellings of some families of trees with a perfect matching:

Theorem 4. *Let T be a tree with a perfect matching and let T' be its contree. If T' is balanced and has an α-labelling, then T also has an α-labelling.*

Corollary 5. *Let T be a tree with a perfect matching such that its contree T' is a balanced caterpillar. Then, T has an α-labelling.*

It is important to remark that the contree of a lobster with a perfect matching is a caterpillar. Thus, by Corollary 5, we obtain the following result.

Corollary 6. *If G is a lobster with a perfect matching such that its contree is balanced, then G has an α-labelling.* □

Theorem 3 and Corollary 6 reinforce Conjecture 1 and points towards an affirmative answer to Question 2. In the next section, we present additional definitions as well as classic results and techniques that are used in our proofs. The proof of Theorem 3 is presented in Section 3 and the proof of Theorem 4 and Corollary 5 are presented in Section 4.

2 Preliminaries

In 1973, Kotzig [7] showed the existence of an α-labelling in a bipartite graph G is equivalent to the existence of a special geometric representation of G he called a 'π-representation'. We will use this representation to prove Theorem 3; we will introduce it in the context of trees.

Let T be a tree with bipartition $\{V_1, V_2\}$. A π-representation of T consists of a drawing of T such that:

(i) the vertices of V_1 are on the line $y = 1$, while those of V_2 are on the line $y = -1$, and consecutive vertices on each of these lines are distance 1 apart;

(ii) the edges of T are straight line segments; and

(iii) if two edges cross, the crossing is not on the line $y = 0$.

Two π-representations of the path P_8 are illustrated in Figure 2.

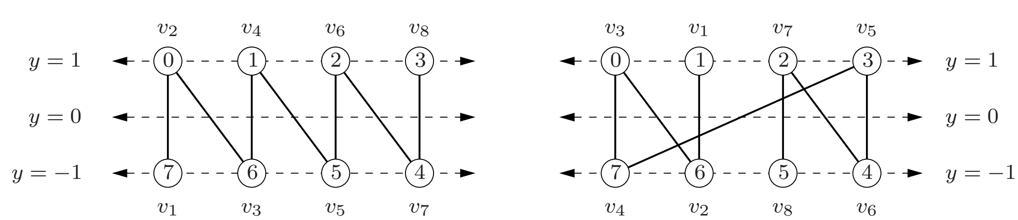


Figure 2: Two π-representations θ and θ' of P_8. In each of these, P_8 is shown with an α-labelling.

The reader should note that the particular three parallel lines used are not relevant, as long as the middle one (i.e., $y = 0$) is half way between the other two. Also, the distance (1 above) between consecutive vertices on the other two lines ($y = 1$ and $y = -1$) is irrelevant, as long as it is constant.

It is important to us that if θ is a π-representation of T, and, for any two real numbers r, s, if, for each $v \in V_1$, we change the x-coordinate of $\theta(v)$ by r and, for each $v \in V_2$, we change the x-coordinate of v by s, then we get another π-representation of T.

Kotzig turns a π-representation of a tree T into an α-labelling as follows: label the leftmost vertex on line $y = 1$ with 0 and continue labelling the vertices consecutively along $y = 1$ until reaching the rightmost vertex on this line, which receives label $|V_1| - 1$; then label the rightmost vertex on line $y = -1$ with $|V_1|$ and continue labelling the vertices consecutively along $y = -1$ until reaching the leftmost vertex on this line, which receives label $|E(T)|$. Figure 2 shows two α-labellings of P_8. Kotzig proved that this is an α-labelling of T and that, conversely, the inverse function converts an α-labelling of T into a π-representation. That is, Kotzig proved that a tree has an α-labelling if and only if it has a π-representation.

Let θ be a π-representation of a tree T with bipartition $\{V_1, V_2\}$. For each vertex v of T, with $v \in V_i$, we let $d^{\leftarrow}_{\theta}(v)$ be the number of vertices of V_i that are to the left of v on the line $y = (-1)^{i-1}$ in θ, while $d^{\rightarrow}_{\theta}(v)$ is the number to the right. Evidently, $d^{\leftarrow}_{\theta}(v) + d^{\rightarrow}_{\theta}(v) = |V_i| - 1$. When the particular π-representation is clear from context, we will drop the subscript θ. As an example, in the π-representation θ of Figure 2, vertex v_1 has $d^{\rightarrow}_{\theta}(v_1) = 0$ and $d^{\leftarrow}_{\theta}(v_1) = 3$, and vertex v_6 has $d^{\rightarrow}_{\theta}(v_6) = 2$ and $d^{\leftarrow}_{\theta}(v_6) = 1$.

Among other things, Kotzig showed how to link π-representations of two trees to get a π-representation of a larger tree. This is our next lemma and is illustrated in Figure 3.

Lemma 7 (Kotzig [7])**.** *Let θ' and θ'' be π-representations of trees T' and T'', respectively, such that there exist $u \in V(T')$ and $v \in V(T'')$ for which $d^{\leftarrow}_{\theta'}(u) = d^{\rightarrow}_{\theta''}(v)$. Then, tree T obtained from $T' \cup T''$ by adding a new edge uv has a π-representation.* □

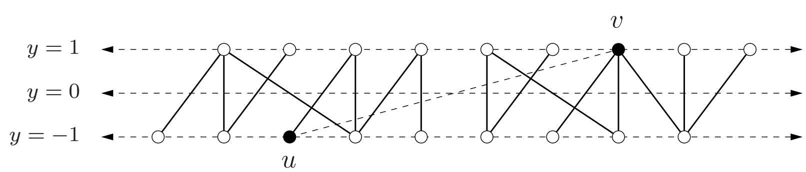
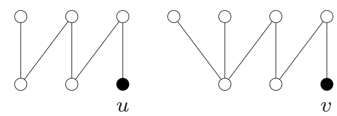


Figure 3: Linking two π-representations θ' and θ'' by an edge uv, $u \in V(T')$ and $v \in V(T'')$. Note that $d^{\leftarrow}_{\theta'}(u) = d^{\rightarrow}_{\theta''}(v) = 2$. Furthermore, the drawing resulting from the addition of edge uv is a π-representation since no other edge cross $y = 0$ at the same point as uv.

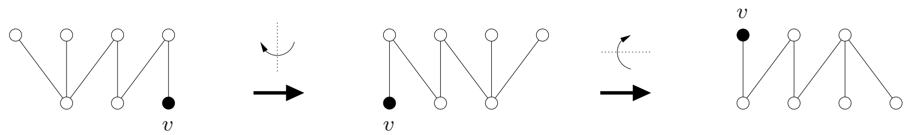
Note that, in order to link two π-representations (one for each of T', T''), we add a straight line segment connecting two vertices $u \in V(T')$ and $v \in V(T'')$ such that $d^{\leftarrow}_{\theta'}(u) = d^{\rightarrow}_{\theta''}(v)$ and that belong to distinct lines $y = 1$ and $y = -1$. If these two conditions are not mutually satisfied, but either $d^{\leftarrow}_{\theta''}(v)$ or $d^{\rightarrow}_{\theta''}(v)$ is equal to $d^{\leftarrow}_{\theta'}(u)$, we can apply one or two reflections to the π-representation θ'' so as to obtain a new π-representation θ''' of T'' such that u and v lie on distinct lines and $d^{\leftarrow}_{\theta'}(u) = d^{\rightarrow}_{\theta'''}(v)$. Figure 4 illustrates this operation.

3 Lobsters with maximum degree three

In this section, we present our main results. Let G be a lobster with $\Delta(G) = 3$, without Y-legs, and with at most one forbidden ending. As previously observed, it suffices to show that G has a π-representation to conclude that G has an α-labelling. In order to construct a π-representation



(a) It is not possible to link u and v since they belong to the same line L_2 and $d^{\leftarrow}_{\theta'}(u) \neq d^{\rightarrow}_{\theta''}(v)$.



(b) We show the horizontal and vertical reflections that can be performed on θ'' so as to obtain a new π-representation θ''' of T'' with $d^{\rightarrow}_{\theta'''}(v) = 0$ and such that v lies on line $y = 1$.

Figure 4: Illustration of the horizontal and vertical reflections that can be performed on a π-representation so as to change the line position of a specific vertex v and the value $d^{\rightarrow}(v)$.

for G, we partition $V(G)$ into subsets $B_1, \ldots, B_k$, find a suitable π-representation for each induced subgraph $G[B_i]$ and, finally, show that these π-representations can be linked in order to obtain a π-representation of the original lobster G.

Let G be a lobster with maximum degree three and without Y-legs. Let $P = (s_1, \ldots, s_t)$ be the spine of G. Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a partition of $V(G)$ into *blocks* B_i such that:

(i) for $1 \leq i \leq k$, $B_i \cap V(P)$ is the non-empty set $\{s_j, \ldots, s_{j'}\}$ of consecutive vertices of P, with $1 \leq j \leq j' \leq t$, and we set $\ell_i = s_j$ and $r_i = s_{j'}$;

(ii) if $1 \leq i < j \leq k$, $r_i = s_p$ and $\ell_j = s_q$, then $p < q$;

(iii) $E(G) = \{\bigcup E(G[B_i])\} \cup \{r_i \ell_{i+1} \colon 1 \leq i \leq k-1\}$, that is, $E(G) \backslash \{\bigcup E(G[B_i])\}$ comprises edges that link consecutive blocks;

Set $\mathcal{B}$ is called a *block-partition* and it is illustrated in Figure 5. Lemma 7 immediately implies the following fact; this is the core of our proof of Theorem 3.

Lemma 8. *Let G be a lobster with maximum degree three and without Y-legs. If G has a block-partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ such that, for $1 \leq i \leq k-1$, subgraphs $G[B_i]$ and $G[B_{i+1}]$ have π-representations θ_i and θ_{i+1}, respectively, such that $d^{\leftarrow}_{\theta_i}(r_i) = d^{\rightarrow}_{\theta_{i+1}}(\ell_{i+1})$, then G has a π-representation.* □

Figure 6 illustrates the application of Lemma 8 for the lobster G of Figure 5.

Let G be a lobster with maximum degree three, without Y-legs and with at most one forbidden ending. Let $P = (s_1, \ldots, s_t)$ be the spine of G and $\mathcal{B} = \{B_1, \ldots, B_k\}$ a block-partition of $V(G)$. We say that block B_i is a:

(i) $\mathcal{C}$-*block*, if no vertex in $B_i \cap V(P)$ is in a 2-leg of G;

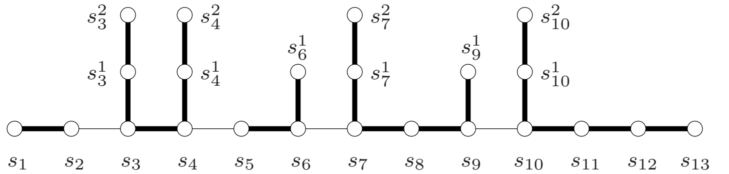
(ii) $\mathcal{L}$-*block*, if

  (a) ℓ_i is in a 2-leg of G;

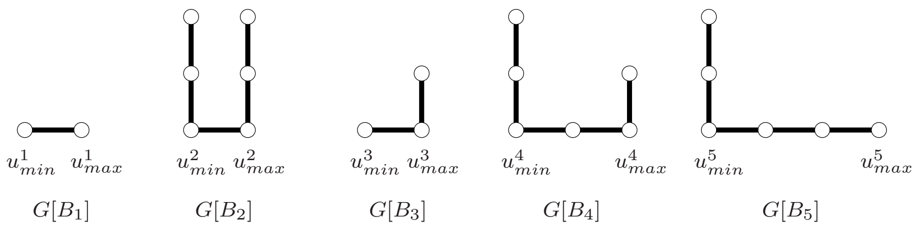
  (b) r_i is in a 1-leg or 2-leg;

  (c) and no vertex in $(B_i \cap V(P)) \backslash \{\ell_i, r_i\}$ is in a 2-leg of G;

(iii) $\mathcal{E}$-*block*, if

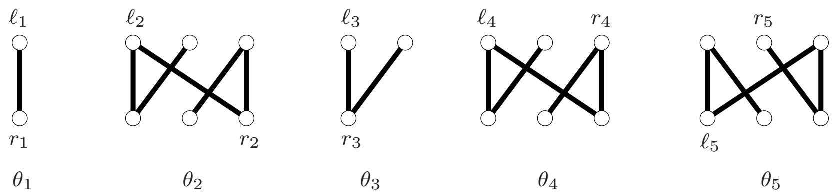


(a) Lobster G. We partition $V(G)$ into blocks $B_1 = \{s_1, s_2\}$, $B_2 = \{s_3, s_4, s_3^1, s_3^2, s_4^1, s_4^2\}$, $B_3 = \{s_5, s_6, s_6^1\}$, $B_4 = \{s_7, s_8, s_9, s_7^1, s_7^2, s_9^1\}$, $B_5 = \{s_{10}, s_{11}, s_{12}, s_{13}, s_{10}^1, s_{10}^2\}$. Thin edges link consecutive blocks B_i and B_{i+1}.

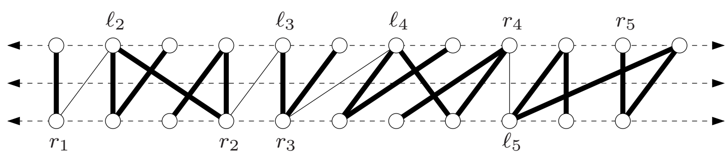


(b) Induced subgraphs $G[B_1]$, $G[B_2]$, $G[B_3]$, $G[B_4]$ and $G[B_5]$.

Figure 5: Block-partition $\mathcal{B}$ of a lobster G with maximum degree three and without Y-legs.



(a) π-representations of subgraphs $G[B_1]$, $G[B_2]$, $G[B_3]$, $G[B_4]$, $G[B_5]$ of Figure 5(b), with $d^{\leftarrow}_{\theta_i}(r_i) = d^{\rightarrow}_{\theta_{i+1}}(l_{i+1}) = 0$.



(b) A π-representation of G obtained by adding edges $r_i l_{i+1}$ to the π-representations of subgraphs $G[B_i]$ and $G[B_{i+1}]$, for $1 \leq i \leq 4$.

Figure 6: Construction of a π-representation for the lobster G of Figure 5.

(a) ℓ_i is in a 2-leg;

(b) and no vertex in $(B_i \cap V(P))\backslash\{\ell_i\}$ is in a 2-leg of G.

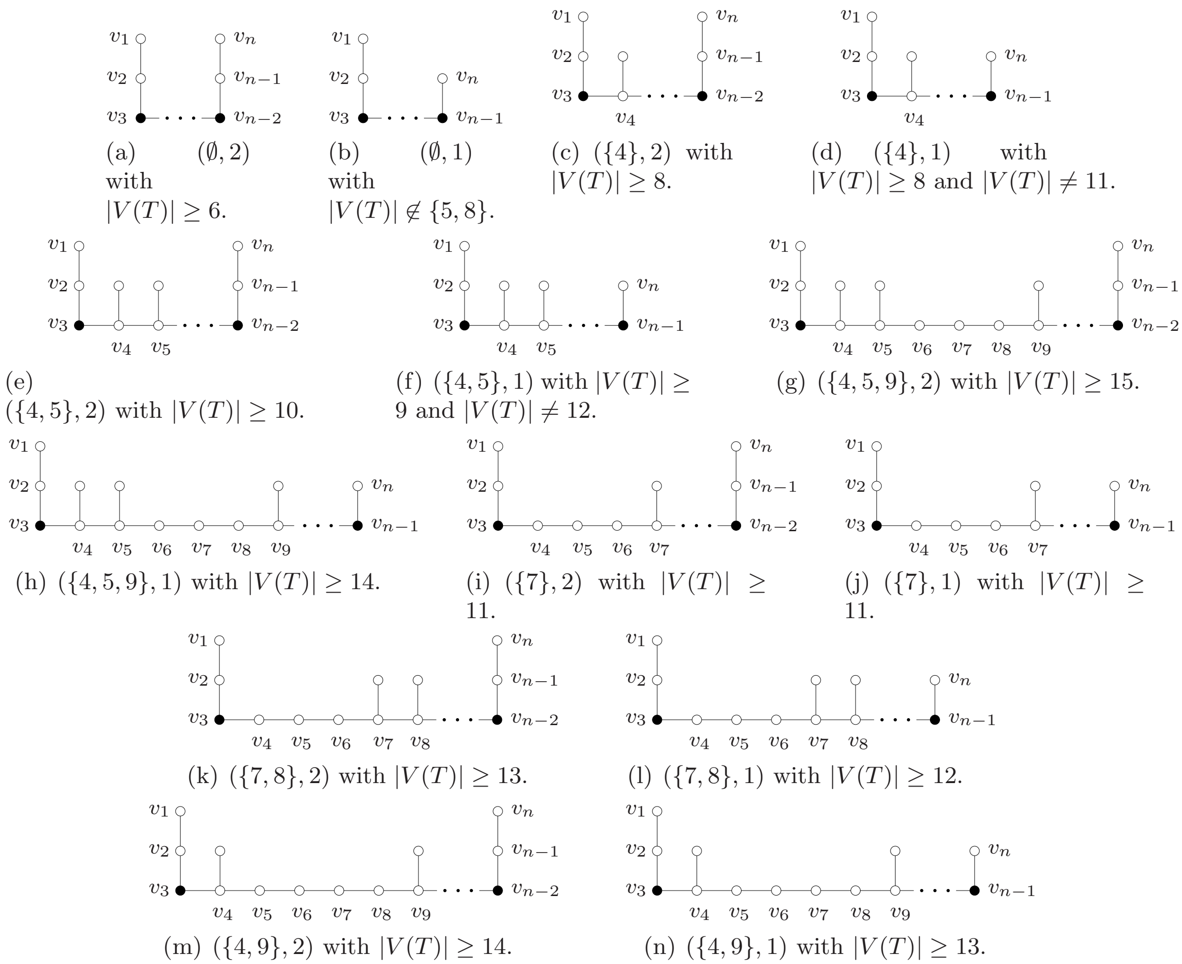
As an example, in Figure 6, blocks B_1 and B_3 are $\mathcal{C}$-blocks, blocks B_2 and B_4 are $\mathcal{L}$-blocks, and block B_5 is an $\mathcal{E}$-block.

Note that, by the definition, the subgraphs induced by $\mathcal{C}$-blocks are caterpillars. Kotzig [7] proved that every caterpillar T with at least two vertices and spine $(v_1, \ldots, v_n)$ has a π-representation θ such that $d^{\rightarrow}_{\theta}(v_1) = d^{\rightarrow}_{\theta}(v_2) = d^{\leftarrow}_{\theta}(v_{n-1}) = d^{\leftarrow}_{\theta}(v_n) = 0$. Moreover, note that a trivial graph has a π-representation θ such that its unique vertex v has $d^{\rightarrow}_{\theta}(v) = d^{\leftarrow}_{\theta}(v) = 0$. Therefore, if B_i is a $\mathcal{C}$-block, then $G[B_i]$ has a π-representation θ_i such that $d^{\rightarrow}_{\theta_i}(\ell_i) = d^{\leftarrow}_{\theta_i}(r_i) = 0$.

Note that the subgraphs induced by $\mathcal{L}$-blocks and $\mathcal{E}$-blocks are also caterpillars. Lemma 9 shows some families of $\mathcal{L}$-blocks and $\mathcal{E}$-blocks have suitable π-representations that are used in the proof of Theorem 3. In order to present these families, we introduce additional notation.

Let P be the spine of a lobster G and $(s_1, \dots, s_t)$ an order of its vertices so that $s_i s_{i+1} \in E(P)$, $1 \le i < t$. For $1 \le i \le j \le t$, let $\langle P^{i,j} \rangle$ be the subgraph of G consisting of the subpath $(s_i, s_{i+1}, \dots, s_j)$ of P, together with all legs having a vertex in $(s_i, s_{i+1}, \dots, s_j)$. Note that an ending is a $\langle P^{i,j} \rangle$ with either $i = 1$ or $j = t$.

Let $B_i \in \mathcal{B}$. Let $T = G[B_i]$ and $(v_1, \dots, v_n)$ be its spine. We say that T belongs to family $(\{i_1, \dots, i_k\}, b)$ if its degree 3 vertices are $v_{i_1}, \dots, v_{i_k}$ such that $3 < i_1 < i_2 < \dots < i_k < n - b$, for $b \in \{1, 2\}$. Note that $\ell_i = v_3$ and $r_i = v_{n-b}$. Figure 7 schematizes fourteen families of the form $(\{i_1, \dots, i_k\}, b)$, that are presented in Lemma 9.



(a) $(\emptyset, 2)$ with $|V(T)| \ge 6$.
(b) $(\emptyset, 1)$ with $|V(T)| \notin \{5, 8\}$.
(c) $(\{4\}, 2)$ with $|V(T)| \ge 8$.
(d) $(\{4\}, 1)$ with $|V(T)| \ge 8$ and $|V(T)| \ne 11$.
(e) $(\{4, 5\}, 2)$ with $|V(T)| \ge 10$.
(f) $(\{4, 5\}, 1)$ with $|V(T)| \ge 9$ and $|V(T)| \ne 12$.
(g) $(\{4, 5, 9\}, 2)$ with $|V(T)| \ge 15$.
(h) $(\{4, 5, 9\}, 1)$ with $|V(T)| \ge 14$.
(i) $(\{7\}, 2)$ with $|V(T)| \ge 11$.
(j) $(\{7\}, 1)$ with $|V(T)| \ge 11$.
(k) $(\{7, 8\}, 2)$ with $|V(T)| \ge 13$.
(l) $(\{7, 8\}, 1)$ with $|V(T)| \ge 12$.
(m) $(\{4, 9\}, 2)$ with $|V(T)| \ge 14$.
(n) $(\{4, 9\}, 1)$ with $|V(T)| \ge 13$.

Figure 7: Schemas for fourteen families of caterpillars $T = (\{i_1, \dots, i_k\}, b)$ that are used in our block-decomposition of a lobster G with $\Delta(G) = 3$, without Y-legs and with at most one forbidden ending. Each member of one of these families is isomorphic to an $\mathcal{L}$-block or an $\mathcal{E}$-block.

Lemma 9. *Let T be a caterpillar with spine $(v_1, \dots, v_n)$ such that T belongs to one of these families:*

(i) $(\emptyset, 2)$ *with* $|V(T)| \ge 6$*;*

(ii) $(\{4\}, 2)$ *with* $|V(T)| \ge 8$*;*

(iii) $(\{4, 5\}, 2)$ *with* $|V(T)| \ge 10$*;*

(iv) $(\{7\}, 2)$ *with* $|V(T)| \geq 11$*;*

(v) $(\{7, 8\}, 2)$ *with* $|V(T)| \geq 13$*;*

(vi) $(\{4, 5, 9\}, 2)$ *with* $|V(T)| \geq 15$*;*

(vii) $(\{4, 9\}, 2)$ *with* $|V(T)| \geq 14$*;*

(viii) $(\emptyset, 1)$ *with* $|V(T)| \neq 5$ *and* $|V(T)| \neq 8$*;*

(ix) $(\{4, 5\}, 1)$ *with* $|V(T)| \geq 9$ *and* $|V(T)| \neq 12$*;*

(x) $(\{4\}, 1)$ *with* $|V(T)| \geq 8$ *and* $|V(T)| \neq 11$*;*

(xi) $(\{7\}, 1)$ *with* $|V(T)| \geq 11$*;*

(xii) $(\{7, 8\}, 1)$ *with* $|V(T)| \geq 12$*;*

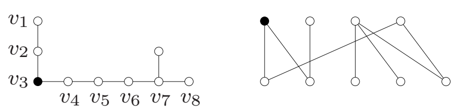
(xiii) $(\{4, 5, 9\}, 1)$ *with* $|V(T)| \geq 14$*;*

(xiv) $(\{4, 9\}, 1)$ *with* $|V(T)| \geq 13$.

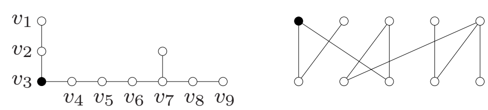
Then, T has a π-representation θ such that $d^{\rightarrow}_{\theta}(v_3) = d^{\leftarrow}_{\theta}(v_{n-b}) = 0$. □

The next lemma provides particular π-representations of a few more caterpillars. The caterpillars and the required π-representations are illustrated in Figure 8, which also constitutes the proof.

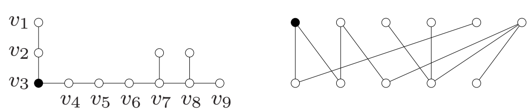
Lemma 10. *Let T be one of the caterpillars presented in Figure 8, and $(v_1, \ldots, v_n)$ be its spine. Then, T is isomorphic to an $\mathcal{E}$-block and has a π-representation θ such that vertex v_3 has $d^{\rightarrow}_{\theta}(v_3) = 0$.* □



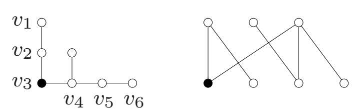
(a) Caterpillar E_1.



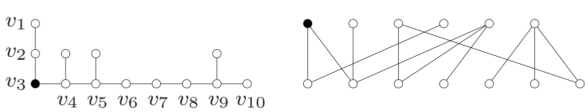
(b) Caterpillar E_2.



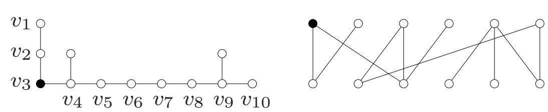
(c) Caterpillar E_3.



(d) Caterpillar E_4.



(e) Caterpillar E_5.



(f) Caterpillar E_6.

Figure 8: Six caterpillars and π-representations of each one of them such that $d^{\rightarrow}(v_3) = 0$.

Now, we are ready to present Lemma 11, which is fundamental for proving Theorem 3.

Lemma 11. *Let G be a lobster with $\Delta(G) = 3$ and without Y-legs. If G has at most one forbidden ending, then there exists a block-partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ of $V(G)$ such that: (i) for $1 \leq i \leq k-1$, subgraph $G[B_i]$ has a π-representation θ_i such that $d^{\rightarrow}_{\theta_i}(\ell_i) = d^{\leftarrow}_{\theta_i}(r_i) = 0$; and (ii) $G[B_k]$ has a π-representation θ_k such that $d^{\rightarrow}_{\theta_k}(\ell_k) = 0$.*

Proof. Let G be a lobster as stated in the hypothesis. Let $P = (s_1, \ldots, s_t)$ be the spine of G. We choose the orientation of P so that s_t is not in a forbidden ending.

Proceeding in order from s_1, we partition $V(G)$ into blocks $B_1, \ldots, B_k$ that are $\mathcal{C}$-blocks, $\mathcal{L}$-blocks or $\mathcal{E}$-blocks. This block-partition of $V(G)$ is constructed inductively. If there is no 2-leg in G, then G is a caterpillar and is the only block; it is a $\mathcal{C}$-block and we are done. Thus, we may assume G has 2-legs. Let j be the least index such that s_j is in a 2-leg; we set the the first block B_1 to be $\langle P^{1,j-1}\rangle$. This is a $\mathcal{C}$-block.

Now, suppose there exists $i \geq 1$ such that the i^{th} block B_i has been determined with $r_i = s_{j-1}$ and such that B_i has a π-representation with $d^{\leftarrow}(r_i) = 0$. In the first case, suppose that there is no $q > j-1$ such that s_q is in a 2-leg. Then, $\langle P^{j,t}\rangle$ is the last block B_{i+1}; it is a $\mathcal{C}$-block and we are done.

In the remaining case, there is a least $q > j-1$ such that s_q is in a 2-leg. If $q > j$, then the next block B_{i+1} is the subgraph $\langle P^{j,q-1}\rangle$, that is, a $\mathcal{C}$-block. Thus, we may assume $q = j$. If there is an $l > j$ such that subgraph $\langle P^{j,l}\rangle$ is an $\mathcal{L}$-block, then, choosing the minimal such l yields one of the graphs in Lemma 9 as B_{i+1} (this claim is proved below). If no such l exists, subgraph $\langle P^{j,t}\rangle$ is the last block B_k, it is an $\mathcal{E}$-block isomorphic to one of the graphs presented in Lemma 9 and Lemma 10 (this claim is proved below).

If B_{i+1} is a $\mathcal{C}$-block or an $\mathcal{L}$-block presented in Lemma 9, then $G[B_{i+1}]$ has a π-representation such that $d^{\rightarrow}(\ell_{i+1}) = 0$ and $d^{\leftarrow}(r_{i+1}) = 0$. Moreover, if B_{i+1} is the last block of the block-partition, either B_{i+1} is a $\mathcal{C}$-block or an $\mathcal{E}$-block isomorphic to one of the graphs presented in Lemma 9 and Lemma 10. In both cases, $G[B_{i+1}]$ has a π-representation in which $d^{\rightarrow}(\ell_{i+1}) = 0$.

Now, we prove the claims above, that is, if the i^{th} block B_i has $r_i = s_{j-1}$ and the next spine vertex s_j is in a 2-leg, then block B_{i+1}, as previously defined, is isomorphic to one of the graphs presented in Lemma 9 and Lemma 10.

If no vertex in $\langle P^{j+1,t}\rangle$ has degree 3 in G, then B_{i+1} is the subgraph $\langle P^{j,t}\rangle$, that is, an $\mathcal{E}$-block $(\emptyset, 2)$, and has $|V(\langle P^{j,t}\rangle)| \geq 6$. Thus, the result follows. Now, assume that there exists a degree 3 vertex in $\{s_{j+1}, \ldots, s_t\}$. Let $\ell = \min\{r \colon r \in \{j+1, \ldots, t\}$ and $d_G(s_r) = 3\}$. Note that $\langle P^{j,\ell}\rangle$ is an $\mathcal{L}$-block. If s_ℓ is in a 2-leg, then $\langle P^{j,\ell}\rangle$ has at least six vertices and is isomorphic to $\mathcal{L}$-block $(\emptyset, 2)$. Therefore, the result follows. For the remaining cases we assume that s_ℓ is in a 1-leg. Let $T = \langle P^{j,\ell}\rangle$. Note that T is an $\mathcal{L}$-block $(\emptyset, 1)$. Thus, if $|V(T)| \neq 5$ and $|V(T)| \neq 8$, the result follows. Now, we consider the cases $|V(T)| = 5$ and $|V(T)| = 8$.

Case 1. $|V(T)| = 5$.

First, suppose no vertex in $\langle P^{\ell+1,t}\rangle$ has degree 3 in G. Let $T' = \langle P^{j,t}\rangle$. Since T' cannot be isomorphic to a forbidden ending, $|V(T')| \geq 7$. If $|V(T')| = 7$, T' is the graph illustrated in Figure 8(d); otherwise, it is isomorphic to an $\mathcal{E}$-block $(\{4\}, 2)$, and the result follows. Now, assume that there exists a degree 3 vertex in $\{s_{\ell+1}, \ldots, s_t\}$. Let $\ell_1 = \min\{r \colon r \in \{\ell+1, \ldots, t\}$ and $d_G(s_r) = 3\}$. Note that $\langle P^{j,\ell_1}\rangle$ is an $\mathcal{L}$-block. If s_{ℓ_1} is in a 2-leg, then $\langle P^{j,\ell_1}\rangle$ has at least 8 vertices and is isomorphic to $\mathcal{L}$-block $(\{4\}, 2)$. Therefore, the result follows. For the remaining cases, assume that s_{ℓ_1} is in a 1-leg.

Let $T_1 = \langle P^{j,\ell_1}\rangle$. Note that T_1 has at least 7 vertices and is an $\mathcal{L}$-block $(\{4\}, 1)$. Thus, if $|V(T_1)| \neq 7$ and $|V(T_1)| \neq 11$, the result follows. Now, we consider the cases $|V(T_1)| = 7$ and $|V(T_1)| = 11$.

Case 1.1. $|V(T_1)| = 7$.

First, suppose no vertex in $\langle P^{\ell_1+1,t}\rangle$ has degree 3 in G. Let $T_1' = \langle P^{j,t}\rangle$. Since T_1' cannot be isomorphic to a forbidden ending, $|V(T_1')| \geq 9$. If $|V(T_1)| = 9$, then T_1' is isomorphic to an $\mathcal{E}$-block

$(\{4,5\},1)$; otherwise, T_1' is isomorphic to an $\mathcal{E}$-block $(\{4,5\},2)$, and the result follows. Now, assume that there exists a degree 3 vertex in $\{s_{\ell_1+1},\ldots,s_t\}$.

Let $\ell_2 = \min\{r\colon r \in \{s_{\ell_1+1},\ldots,t\}$ and $d_G(s_r)=3\}$. Note that $\langle P^{j,\ell_2}\rangle$ is an $\mathcal{L}$-block. If s_{ℓ_2} is in a 2-leg, then $\langle P^{j,\ell_2}\rangle$ has at least 10 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,5\},2)$. Therefore, the result follows. For the remaining cases we assume that s_{ℓ_2} is in a 1-leg. Let $T_2 = \langle P^{j,\ell_2}\rangle$. Note that T_2 has at least 9 vertices and is an $\mathcal{L}$-block $(\{4,5\},1)$. Thus, if $|V(T_2)| \neq 12$, the result follows.

Now, consider $|V(T_2)| = 12$. First, suppose no vertex in $\langle P^{\ell_2+1,t}\rangle$ has degree 3 in G. Let $T_2' = \langle P^{j,t}\rangle$. Note that $|V(T_2')| \geq 13$. If $|V(T_2')| = 13$, T_2' is the graph illustrated in Figure 8(e); otherwise, T_2' is isomorphic to an $\mathcal{E}$-block $(\{4,5,9\},1)$ and the result follows. Now, assume that there exists a degree 3 vertex in $\{s_{\ell_2+1},\ldots,s_t\}$. Let $\ell_3 = \min\{r\colon r \in \{\ell_2+1,\ldots,t\}$ and $d_G(s_r)=3\}$. Note that $\langle P^{j,\ell_3}\rangle$ is an $\mathcal{L}$-block. If s_{ℓ_3} is in a 2-leg, then $\langle P^{j,\ell_3}\rangle$ has at least 15 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,5,9\},2)$. If s_{ℓ_3} is in a 1-leg, then $\langle P^{j,\ell_3}\rangle$ has at least 14 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,5,9\},1)$. In both cases, the result follows.

Case 1.2. $|V(T_1)| = 11$.

First, suppose no vertex in $\langle P^{\ell_1+1,t}\rangle$ has degree 3 in G. Let $T_1' = \langle P^{j,t}\rangle$. Note that $|V(T_1')| \geq 12$. If $|V(T_1')| = 12$, T_1' is the graph illustrated in Figure 8(f); otherwise, T_1' is isomorphic to an $\mathcal{E}$-block $(\{4,9\},1)$ with $|V(T_1')| \geq 13$ and the result follows.

Now, assume that there exists a degree 3 vertex in $\{s_{\ell_1+1},\ldots,s_t\}$. Let $\ell_2 = \min\{r\colon r \in \{\ell_1+1,\ldots,t\}$ and $d_G(s_r)=3\}$. Note that $\langle P^{j,\ell_2}\rangle$ is an $\mathcal{L}$-block. If s_{ℓ_2} is in a 2-leg, then $\langle P^{j,\ell_2}\rangle$ has at least 14 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,9\},2)$. If s_{ℓ_2} is in a 1-leg, then $\langle P^{j,\ell_2}\rangle$ has at least 13 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,9\},1)$. In both cases, the result follows.

Case 2. $|V(T)| = 8$.

First, suppose no vertex in $\langle P^{\ell+1,t}\rangle$ has degree 3 in G. Let $T' = \langle P^{j,t}\rangle$. Note that $|V(T')| \geq 9$. If $|V(T')| \in \{9,10\}$, T' is exhibited in Figure 8(a) and Figure 8(b); otherwise, it is isomorphic to an $\mathcal{E}$-block $(\{7\},1)$ and the result follows. Now, assume that there exists a degree 3 vertex in $\{s_{\ell+1},\ldots,s_t\}$. Let $\ell_1 = \min\{r\colon r \in \{\ell+1,\ldots,t\}$ and $d_G(s_r)=3\}$. Note that $\langle P^{j,\ell_1}\rangle$ is an $\mathcal{L}$-block. If s_{ℓ_1} is in a 2-leg, then $\langle P^{j,\ell_1}\rangle$ has at least 11 vertices and is isomorphic to $\mathcal{L}$-block $(\{7\},2)$. Therefore, the result follows. For the remaining cases, assume that s_ℓ is in a 1-leg. Let $T_1 = \langle P^{j,\ell_1}\rangle$. Note that T_1 has at least 10 vertices and is an $\mathcal{L}$-block $(\{7\},1)$. Thus, if $|V(T_1)| \geq 11$, the result follows.

Next, consider the case $|V(T_1)| = 10$. First, suppose no vertex in $\langle P^{\ell_1+1,t}\rangle$ has degree 3 in G. Let $T_1' = \langle P^{j,t}\rangle$. Note that $|V(T_1')| \geq 11$. If $|V(T_1')| = 11$, T_1' is the graph illustrated in Figure 8(c); otherwise, T_1' is isomorphic to an $\mathcal{E}$-block $(\{7,8\},1)$, and the result follows.

Now, assume that there exists a degree 3 vertex in $\{s_{\ell_1+1},\ldots,s_t\}$. Let $\ell_2 = \min\{r\colon r \in \{\ell_1+1,\ldots,t\}$ and $d_G(s_r)=3\}$. Note that $\langle P^{j,\ell_2}\rangle$ is an $\mathcal{L}$-block. If s_{ℓ_2} is in a 2-leg, then $\langle P^{j,\ell_2}\rangle$ has at least 13 vertices and is isomorphic to $\mathcal{L}$-block $(\{7,8\},2)$. If s_{ℓ_2} is in a 1-leg, then $\langle P^{j,\ell_2}\rangle$ has at least 12 vertices and is isomorphic to $\mathcal{L}$-block $(\{7,8\},1)$. In both cases, the result follows. This case concludes the proof. $\square$

Now we are ready to prove Theorem 3.

Theorem 3. *Let G be a lobster with $\Delta(G)=3$ and without Y-legs. If G has at most one forbidden ending, then G has an α-labelling.*

Proof. By Lemma 11 and Lemma 8 we conclude that every lobster G with maximum degree three, without Y-legs and with at most one forbidden ending has a π-representation. This implies that G has an α-labelling. $\square$

4 Strongly-α labellings of trees with a perfect matching

In this section, we prove Theorem 4 and Corollary 5. Let T be a tree with a perfect matching M. Remember that the contree of T is the tree obtained from T by contracting the edges of M. A labelling f of T is *strongly-graceful* if f is graceful and, additionally, for each edge $uv \in M$, $f(u) + f(v) = |E(T)|$.

Let f be a strongly-graceful labelling of a tree T with a perfect matching M. There are some properties that arise directly from the definition of strongly-graceful labellings. For instance, for every edge $uv \in M$, the induced label of uv is $|f(u) - f(v)| = |f(u) - (|E(T)| - f(u))| = |2f(u) - |E(T)||$, which is an odd number since $|E(T)|$ is odd. Thus, the parities of the endpoints of each edge in M are different. Moreover, since f is a graceful labelling, the labels of the edges in $E(T)\backslash M$ are the even numbers in set $\{1, \ldots, |E(T)|\}$. Therefore, the endpoints of each edge in $E(T)\backslash M$ have the same parity. These observations are summarized in Proposition 12.

Proposition 12. *Let f be a strongly-graceful labelling of a tree T with a perfect matching M. Then, $f(u) \not\equiv f(v) \pmod 2$, if $uv \in M$ and $f(u) \equiv f(v) \pmod 2$, otherwise. Moreover, $\mathcal{L}^f_M = \{2i + 1 \colon 0 \leq i \leq \lfloor |E(T)|/2 \rfloor\}$ and $\mathcal{L}^f_{E(T)\backslash M} = \{2i \colon 1 \leq i \leq \lfloor |E(T)|/2 \rfloor\}$.* □

Strongly-graceful labellings were introduced by Broersma and Hoede [5]. In their seminal article, the authors showed an equivalence between graceful and strongly-graceful labellings and they proved that all trees are graceful if and only if all trees with a perfect matching have a strongly-graceful labelling. Additionally, the authors proved the following result.

Theorem 13 (Broersma and Hoede [5]). *If the contree of a tree T with a perfect matching has a graceful labelling, then T has a strongly-graceful labelling.* □

Note that the contree of a lobster with a perfect matching is a caterpillar. Since every caterpillar has an α-labelling (which is also a graceful labelling), by Theorem 13, we obtain that every lobster with a perfect matching has a strongly-graceful labelling. While analysing strongly-graceful labellings of lobsters with a perfect matching, we observed that some strongly-graceful labellings are also α-labellings. This observation led us to the concept of strongly-α labellings: we say that a labelling f of T is *strongly-α* if f is strongly-graceful and, additionally, f is also an α-labelling.

While every lobster with a perfect matching has a strongly-graceful labelling, there are lobsters that do not have a strongly-α labelling. Figure 9 exhibits an example.

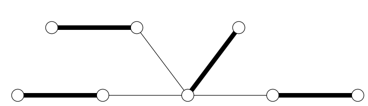
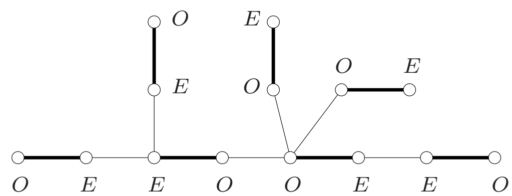


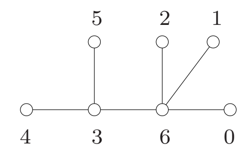
Figure 9: A lobster with a perfect matching that has no strongly-α-labelling.

In Theorem 14, we characterize the trees with a perfect matching that have a strongly-α labelling. In our proof, we use a construction defined by Broersma and Hoede, described below. This construction allows us to obtain a strongly-graceful labelling of T from any graceful labelling of its contree.

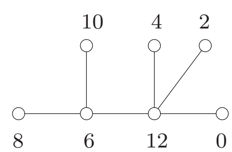
Broersma-Hoede's construction. Let T be a tree with a perfect matching M. By Proposition 12, in a strongly-graceful labelling f of T, we have $f(u) \not\equiv f(v) \pmod 2$, if $uv \in M$, and $f(u) \equiv f(v) \pmod 2$, otherwise. Note that, once the parity of the label of one vertex of T is known, the parities of the other vertices are uniquely determined. This occurs because T has only one perfect matching and there is only one path connecting any two vertices of T. Thus, the first step of the construction is to choose the parity of the label of an arbitrary vertex x of T and, then, obtain the parity of the labels of the remaining vertices. This step is illustrated in Figure 10(a). Next, consider a graceful labelling f' of the contree T' of T. Modify f' so that, for each vertex $x_{uv} \in V(T')$, x_{uv} is assigned the label $2f'(x_{uv})$, as illustrated in Figure 10(b) and Figure 10(c). Let $uv \in M$. Considering that v has even parity and u has odd parity, assign label $2f'(x_{uv})$ to v and label $|E(T)|-2f'(x_{uv})$ to u. Broersma and Hoede proved that this assignment is a strongly-graceful labelling of T.



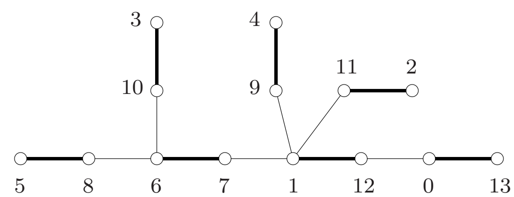
(a) A tree T with a perfect matching. Each vertex of T is labelled with letters O or E, where letter O means odd parity and letter E means even parity. For each edge $uv \in E(T)$, these parities respect the properties stated in Proposition 12.



(b) Contree T' of tree T with a graceful labelling f'.



(c) A new labelling of T' obtained by assigning label $2f'(v)$ to each vertex v.



(d) Strong-graceful labelling f of T. The endpoints of each edge $uv \in M$ have labels with distinct parities. The even label $f(v)$ is taken from the previous labelling of T' and the odd label is $|E(T)| - f(v)$.

Figure 10: Construction of a strong-graceful labelling f for a tree T with a perfect matching. Note that, in this case, f is also an α-labelling.

The classical *complementary labelling* $\overline{f}$ of a graceful labelling f of a graph G is defined by $\overline{f}(v) = |E(G)| - f(v)$, for each vertex $v \in V(G)$. It is immediate that $\overline{f}$ is graceful. One illustration is given in Figure 11.

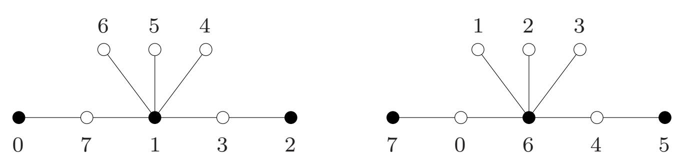


Figure 11: A graceful labelling f of a caterpillar and its complementary labelling.

Now we are ready to prove Theorem 14. Theorem 4 and Corollary 5 follow as consequences of Theorem 14.

Theorem 14. *Let T be a tree with a perfect matching. Then, T has a strongly-α labelling if and only if its contree has a balanced bipartition and an α-labelling.*

Proof. Let T be a tree with a perfect matching M and let T' be the contree of T. Let $n_T = |V(T)|$ and $n_{T'} = |V(T')| = n_T/2$. The result is trivial for $n_T = 2$. Thus, consider $n_T \in \{4p, 4p+2\}$, $p \in \mathbb{N}^*$.

Since T has a perfect matching, its bipartition $\{X, Y\}$ satisfies $|X| = |Y|$. Because f is an α-labelling, $f(X)$ is either $\{0, \ldots, \frac{n_T}{2} - 1\}$ or $\{\frac{n_T}{2}, \ldots, n_T - 1\}$ and $f(Y)$ is the other. We use labelling f so as to obtain an α-labelling g for T'. Let $v_{xy} \in V(T')$ be obtained from T by the contraction of edge $xy \in M$. Proposition 12 implies that $f(x) \not\equiv f(y) \pmod 2$. Let $f'(v_{xy})$ be the one of $f(x)$ and $f(y)$ that is even. Now, Proposition 12 shows that $f' \colon V(T') \to \{0, 2, \ldots, n_T - 2\} = \mathcal{L}^f_{E(T)\backslash M} \cup \{0\}$.

Let $v_{xy}v_{zw} \in E(T')$ with $x, z \in X$ and $y, w \in Y$. By the definition of T', exactly one of xw and zy belongs to $E(T)\backslash M$. Suppose $xw \in E(T)\backslash M$. Note that this implies that $f(x) \equiv f(w) \pmod 2$. Also, if $f(x) \equiv 0 \pmod 2$, then $|f'(v_{xy}) - f'(v_{zw})| = |f(x) - f(w)|$; otherwise, $f(x) \equiv 1 \pmod 2$ and $|f'(v_{xy}) - f'(v_{zw})| = |f(y) - f(z)| = |(n_T - 1 - f(x)) - (n_T - 1 - f(w))| = |f(x) - f(w)|$. We conclude that $\mathcal{L}^{f'}_{E(T')} = \mathcal{L}^f_{E(T)\backslash M}$. For $v \in V(T')$, define $g(v) = f'(v)/2$. By the definition of g, $\mathcal{L}^g_{V(T')} = \{0, \ldots, \frac{n_T}{2} - 1\} = \{0, \ldots, n_{T'} - 1\}$ and $\mathcal{L}^g_{E(T')} = \{1, \ldots, \frac{n_T}{2} - 1\} = \{1, \ldots, n_{T'} - 1\}$. Thus, g is graceful.

Now, we prove that g is an α-labelling. In order to do this, we show that there exists $k \in \mathcal{L}^g_{V(T')}$ such that, either $g(v_{xy}) \leq k < g(v_{zw})$ or $g(v_{zw}) \leq k < g(v_{xy})$, for every edge $v_{xy}v_{zw} \in E(T')$. By the definition of f, $f(x) \leq n_T/2 - 1 < f(w)$. First, suppose that $f(x)$ and $f(w)$ are both even. Thus, we have

$$\begin{aligned} &f(x) \leq n_T/2 - 1 < f(w), \\ &f'(v_{xy}) \leq n_T/2 - 1 < f'(v_{zw}), \\ &g(v_{xy}) \leq n_T/4 - 1/2 < g(v_{zw}). \end{aligned} \tag{1}$$

Now, assume $f(x)$ and $f(w)$ are both odd. Thus, we have

$$\begin{aligned} &f(x) \leq n_T/2 - 1 < f(w), \\ &n_T - 1 - f(x) \geq n_T/2 > n_T - 1 - f(w), \\ &f(y) \geq n_T/2 > f(z), \\ &f'(v_{xy}) \geq n_T/2 > f'(v_{zw}), \\ &g(v_{xy}) \geq n_T/4 > g(v_{zw}). \end{aligned} \tag{2}$$

Hence, if $n_T = 4p$, then let $k = p - 1$; otherwise, $n_T = 4p + 2$ and we let $k = p$. In both cases, $g(v_{xy}) \leq k < g(v_{zw})$ or $g(v_{zw}) \leq k < g(v_{xy})$. In order to conclude the proof, just observe that when $n_{T'} = 2p$, T' has a bipartition with parts of equal size since $k = p - 1$. Moreover, when $n_{T'} = 2p + 1$, T' has a bipartition in which the cardinalities of the parts differ by one since $k = p$. Therefore, T' has a balanced bipartition and the result follows.

Now, suppose T' is balanced and that $g \colon V(T') \to \{0, \ldots, n_{T'} - 1\}$ is an α-labelling. Let $\{A, B\}$ be the bipartition of T', labelled so that $|A| \geq |B|$. Changing to the complementary labelling if necessary, we may assume $\mathcal{L}^g_A = \{0, \ldots, |A| - 1\}$ and $\mathcal{L}^g_B = \{|A|, \ldots, n_{T'} - 1\}$. Since T' has α-labelling g, by Theorem 13, T has a strongly-graceful labelling f obtained by the Broersma-Hoede's construction.

Next, we show that f is also an α-labelling; that is, we prove that there exists an integer $k \in \{0, \ldots, n_T - 1\}$ such that, for each edge $uv \in E(T)$, either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$. We claim that $k = 2|A| - 1$ when $|A| = |B|$ and that $k = 2|A| - 2$ when $|A| = |B| + 1$. Thus, let $k_1 = 2|A| - 1$, $k_2 = 2|A| - 2$ and consider an edge $uv \in E(T)$. There are two cases to analyse depending on which set, M or $E(T)\backslash M$, edge uv belongs to.

Case 1. $uv \in M$.

By the construction of f, vertices u and v receive different labels $2q$ and $(n_T - 1) - 2q$, for $q \in \mathcal{L}_A^g \cup \mathcal{L}_B^g$. Without loss of generality, assume that $f(u) = \min\{2q, (n_T - 1) - 2q\}$ and $f(v) = \max\{2q, (n_T - 1) - 2q\}$.

First, suppose that $q \in \mathcal{L}_A^g$. In this case, $2q < (n_T - 1) - 2q$. Thus, $f(u) = 2q$ and $f(v) = (n_T - 1) - 2q$. Moreover, $f(u) = 2q \leq 2|A| - 2$ and $f(v) \geq (n_T - 1) - (2|A| - 2) = (2n_{T'} - 1) - (2|A| - 2) = (2(|A| + |B|) - 1) - 2|A| + 2 = 2|B| + 1$. Therefore, since $f(u) \leq 2|A| - 2$ and $f(v) \geq 2|B| + 1$, we have:

$$\text{if } |A| = |B|, \text{ then } f(u) \leq 2|A| - 2 < 2|A| - 1 = k_1 < 2|A| + 1 \leq f(v); \tag{3}$$
$$\text{if } |A| = |B| + 1, \text{ then } f(u) \leq 2|A| - 2 = k_2 < 2|A| - 1 \leq f(v). \tag{4}$$

Now, suppose that $q \in \mathcal{L}_B^g$. In this case, $2q > (n_T - 1) - 2q$. Thus, $f(v) = 2q$ and $f(u) = (n_T - 1) - 2q$. Moreover, $f(v) = 2q \geq 2|A|$ and $f(u) \leq (n_T - 1) - 2|A| = (2n_{T'} - 1) - 2|A| = (2(|A| + |B|) - 1) - 2|A| = 2|B| - 1$. Therefore, since $f(u) \leq 2|B| - 1$ and $f(v) \geq 2|A|$, we have:

$$\text{if } |A| = |B|, \text{ then } f(u) \leq 2|A| - 1 = k_1 < 2|A| \leq f(v); \tag{5}$$
$$\text{if } |A| = |B| + 1, \text{ then } f(u) \leq 2|A| - 3 < 2|A| - 2 = k_2 < 2|A| \leq f(v). \tag{6}$$

Case 2. $uv \in E(T) \backslash M$.

By the construction of f, $f(u) \equiv f(v) \pmod 2$. Without loss of generality, assume that $f(u) < f(v)$. First, suppose that $f(u)$ and $f(v)$ are both even. In this case, $f(u) = 2q$ and $f(v) = 2r$, for $q, r \in \mathcal{L}_A^g \cup \mathcal{L}_B^g$. Since $uv \notin M$, edge uv has a corresponding edge $u'v'$ in the contree T' whose endpoints are in different parts of $\{A, B\}$. Since $f(u) < f(v)$, we have that $q < r$ with $q \in \mathcal{L}_A^g$ and $r \in \mathcal{L}_B^g$. Also, since $\mathcal{L}_A^g = \{0, \ldots, |A| - 1\}$ and $\mathcal{L}_B^g = \{|A|, \ldots, n_{T'} - 1\}$, we have that $f(u) = 2q \leq 2|A| - 2$ and $f(v) = 2r \geq 2|A|$. These inequalities imply that $f(u) \leq 2|A| - 2 = k_2 < k_1 < 2|A| \leq f(v)$, and the result follows.

Now, suppose that $f(u)$ and $f(v)$ are both odd. By the construction of f, we have that $f(u) = (n_T - 1) - 2q$ and $f(v) = (n_T - 1) - 2r$, for $q, r \in \mathcal{L}_A^g \cup \mathcal{L}_B^g$. Since $f(u) < f(v)$, $r \in \mathcal{L}_A^g$ and $q \in \mathcal{L}_B^g$. This implies that $2q \geq 2|A|$ and $2r \leq 2|A| - 2$. Since $n_T = 2n_{T'} = 2(|A| + |B|)$, we obtain that $f(u) = (n_T - 1) - 2q = 2|A| + 2|B| - 1 - 2q \leq 2|A| + 2|B| - 1 - 2|A| = 2|B| - 1$ and $f(v) = (n_T - 1) - 2r = 2|A| + 2|B| - 1 - 2r \geq 2|A| + 2|B| - 1 - 2|A| + 2 = 2|B| + 1$. Since $f(u) \leq 2|B| - 1$ and $f(v) \geq 2|B| + 1$, we have that

$$\text{if } |A| = |B|, \text{ then } f(u) \leq 2|A| - 1 = k_1 < 2|A| + 1 \leq f(v); \tag{7}$$
$$\text{if } |A| = |B| + 1, \text{ then } f(u) \leq 2|A| - 3 < k_2 < 2|A| - 1 \leq f(v); \tag{8}$$

and the result follows. □

Theorem 4. *Let T be a tree with a perfect matching and let T' be its contree. If T' has a balanced bipartition and an α-labelling, then T has an α-labelling.* □

Corollary 5. *Let T be a tree with a perfect matching such that its contree T' is a caterpillar with a balanced bipartition. Then T has an α-labelling.*

Proof. The result follows by Theorem 14 and by the fact that every caterpillar has an α-labelling. □

5 Concluding Remarks

In this work, we have considered lobsters that have at most one forbidden ending and lobsters whose contrees are balanced. These are positive steps towards settling Conjecture 1 for lobsters. One possible next step to pursue is to consider lobsters for which neither Theorem 3 nor Corollary 6 apply, as in the example shown in Figure 12.

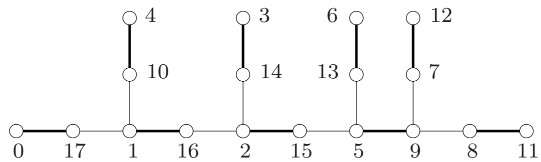


Figure 12: An α-labelling of a lobster G with maximum degree 3 and a perfect matching. Note that G has two forbidden endings and its contree is not balanced.

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