Degree-dependent intervertex separation in complex networks

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(Dated:)

We find the mean length $\ell(k)$ of the shortest paths between a vertex of degree k and other vertices in growing networks. In scale-free networks, we obtain a power-law correction to a logarithmic dependence, $\ell(k) = A\ln[N/k^{(\gamma-1)/2}] - Ck^{\gamma-1}/N + \ldots$. Here N is the number of vertices in the network, γ is the degree distribution exponent, and the coefficients A and C depend on a network. We obtain this result for a number of growing deterministic graphs but believe that it holds for a wide class of evolving scale-free networks. In contrast, in stochastic and deterministic growing trees with an exponential degree distribution, we find a linear dependence on degree, $\ell(k) \cong A\ln N - Ck$. We compare our results for growing networks with those for uncorrelated graphs.

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I. INTRODUCTION

The main objects of interest of the physics of complex networks [1, 2, 3, 4, 5, 6] are extremely compact, infinite dimensional nets—so called small worlds. The basic measure of the compactness of a network is the mean intervertex distance or the mean intervertex separation, that is, the mean length of the shortest path between a pair of vertices, ℓ. (The path runs along edges, each edge has the unit length.) Physicists often use another term for this characteristic—the diameter of a network, although in graph theory the term network diameter is reserved for the maximal separation of a pair of vertices in a net.

A network shows the small-world effect if its mean intervertex distance slowly increases with the network size (the total number of vertices in a network, N), slower than any power-law function of N. This is in contrast to finite dimensional objects, where the mean intervertex distance grows as $N^{1/d}$, d being the dimension of an object. (We discuss sparse networks.) By definition, a small world is a network with the small-world effect. Note that this definition is not related to the presence of loops in a network. Small worlds may be loopy or clustered networks, or they may be without loops—trees.

The mean intervertex distances in networks were extensively studied both in the framework of empirical research [7] and analytically [8, 9, 10, 11]. The typical size dependence of the mean intervertex separation is logarithmic, $\ell(N) \propto \ln N$. However, the mean intervertex distance is an integrated, coarse characteristic. One may be interested in a more delicate issue—the position of an individual vertex in a network. One should note that recently Holyst *et al.* [12], have considered the question: how far are vertices of specific degrees from each other? In the present paper we present results for another (though related) characteristic of a complex network—the mean length of the shortest paths from a vertex of a given degree k to the remaining vertices of the network, $\ell(k)$. In simple terms, we reveal the smallness of a network from the point of view of its vertex of a given degree. We obtain nontrivial dependences $\ell(k)$ for networks with power-law and exponential degree distributions. We mostly consider growing networks, where correlations between the degrees of vertices are important, but for comparison, also discuss uncorrelated networks. In our study we use convenient deterministic growing graphs and compare some of our results with simulations of a stochastic model of a growing network.

In Sec. II we list our results. Section III contains a discussion of the $\ell(k)$ dependence in uncorrelated networks. Section IV explains in detail how the results were obtained.

II. RESULTS

Our results were obtained by using simple deterministic graphs which allow exact solution of the problem. Deterministic small worlds were considered in a number of recent papers [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and have turned out to be a useful tool. (We called these networks *pseudofractals*. Indeed, at first sight, they look as fractals. However, they are infinite dimensional objects, so that they are not fractals.) These graphs correctly reproduce practically all known network characteristics. We use a set of deterministic scale-free models with various values of the degree distribution exponent γ, $P(k) \propto k^{-\gamma}$ (see Fig. 1). We consider deterministic graphs with γ in the range between 2 and ∞, where a graph with $\gamma = \infty$ has an exponentially decreasing (discrete) spectrum of degrees.

In the scale-free deterministic graphs, the result for the mean separation of a vertex of degree k from the

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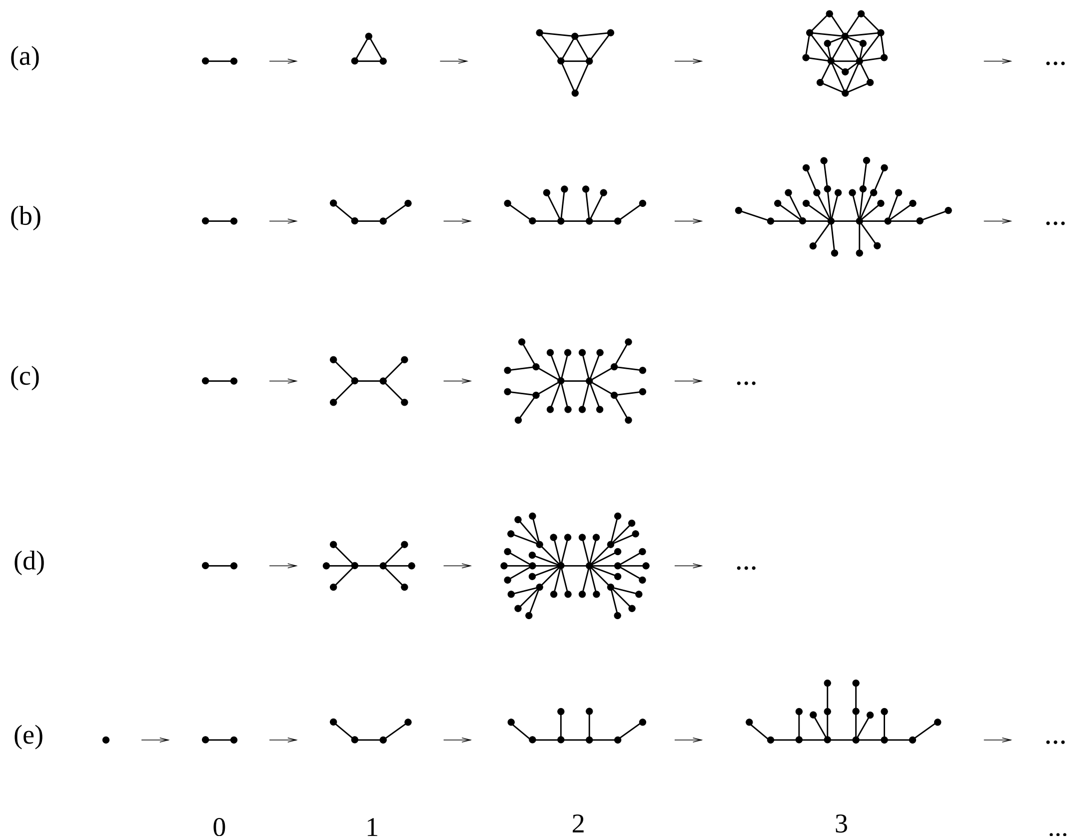

FIG. 1: The set of deterministic graphs that is used in this paper. (a) A scale-free graph with the exponent of the degree distribution $\gamma = 1 + \ln 3/\ln 2 = 2.585\ldots$ [2, 14]. At each step, each edge of the graph transforms into a triangle. (b) A scale-free tree graph with $\gamma = 1 + \ln 3/\ln 2 = 2.585\ldots$ [15]. At each step, a pair of new vertices is attached to the ends of each edge of the graph. (c) A scale-free tree graph with $\gamma = 3$. At each step, a pair of new vertices is attached to the ends of each edge plus a new vertex is attached to each vertex of the graph. (d) A scale-free tree graph with $\gamma = 1 + \ln 5/\ln 2 = 3.322\ldots$. At each step, a pair of new vertices is attached to the ends of each edge plus two new vertices are attached to each vertex of the graph. (e) A deterministic graph with an exponentially decreasing spectrum of degrees [15]. At each step, a new vertex is attached to each vertex of the graph. In all these graphs, a mean intervertex distance grows with the number N of vertices as $\ln N$.

remaining vertices of the network looks as follows:

$$\ell(k) = A\,\ln\!\left[\frac{N}{k^{(\gamma-1)/2}}\right] - C\,\frac{k^{\gamma-1}}{N} + \ldots\,. \tag{1}$$

The constants A and C (and the sign of C) depend on a particular network. This asymptotic formula is true for large enough k. We believe that this dependence is valid for a wide class of random, growing, scale-free networks. One should note that in all of the growing networks considered in this paper, new connections cannot emerge between already existing vertices. These networks are often called "citation graphs". For more general scale-free graphs, one may suggest the main contribution of the form $\ell(k) \cong A \ln N + B \ln k$.

In the specific point $\gamma = 3$, correlations between the degrees of the nearest neighbors in these graphs are anomalously low. In this situation, the main contribution to $\ell(k)$ reduces to $\ell(k) \propto \ln(N/k)$, which coincides with the result for equilibrium uncorrelated networks (see the next section).

Formula (1) fails at $\gamma \to \infty$. E.g., it cannot be applied for networks with an exponential degree distribution. For growing trees with this distribution, the resulting dependence turned out to be

$$\ell(k) \cong A \ln N - Ck\,, \tag{2}$$

where the constants A and C depend on a network. We obtained this dependence analytically for deterministic graphs (trees) with an exponential degree distribution [e.g., graph (e) in Fig. 1]. We observed the same dependence in a stochastically growing tree with random attachment. In this tree (with an exponential degree distribution), at each time step, a new vertex is attached to a randomly selected vertex of the net. The result of the simulation of this network is shown in Fig. 2(a). In both the networks—graph (e) in Fig. 1 and the corresponding stochastic net with random attachment—the slope of the degree dependence equals $-1/2$. More generally, if in a growing tree of this kind, at each step, n new vertices become attached to a vertex, the slope of the degree dependence equals $-1/(n+1)$ [see Fig. 2(b)].

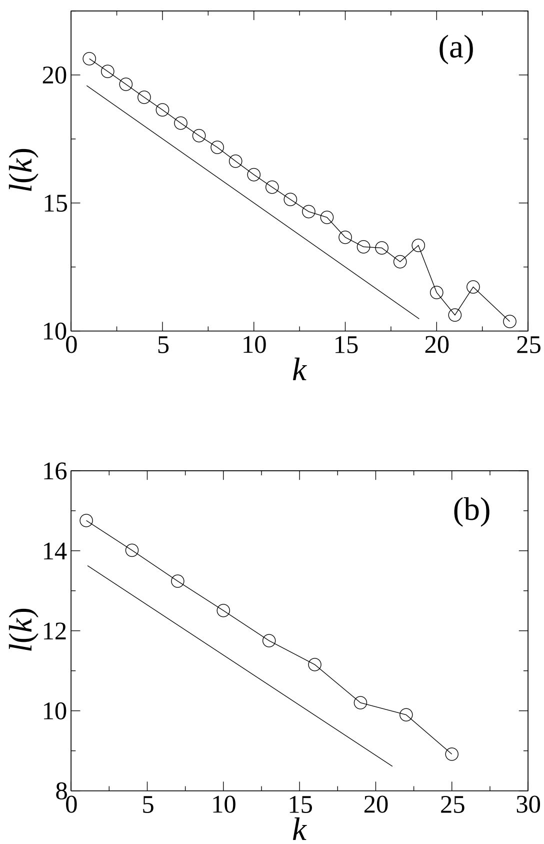

FIG. 2: Degree-dependent mean intervertex separation of stochastic networks (trees) growing under the mechanism of random attachment. These networks have exponential degree distributions. (a) At each time step, a vertex is attached to a randomly chosen vertex of the network. The dependence is the result of the simulation of the network of 10^5 vertices, 50 runs. For comparison, a line with a slope $-1/2$ is shown. (b) At each time step, 3 vertices are attached to a randomly chosen vertex of the network. The dependence is presented for the network of 9998 vertices, 50 runs. For comparison, a line with a slope $-1/4$ is shown.

All networks that we studied, had the generic property:

$$\max_k \ell(k) = 2 \min_k \ell(k) \,, \tag{3}$$

in the large network limit. As is natural, the maximum value of $\ell(k)$ is realized at the minimal degree of a vertex in a network, and vise versa, the minimum value of $\ell(k)$ is attained at the maximum degree.

III. $\ell(k)$ OF AN UNCORRELATED NETWORK

The configuration model [24, 25, 26, 27] is a standard model of an uncorrelated (equilibrium) random network. In simple terms, these are maximally random graphs with a given degree distribution. In the large network limit, they have relatively few loops and almost surely are trees in any local environment of a given vertex. The mean intervertex distance ℓ in these networks is estimated in the following way, Ref. [8] (see also Refs. [9, 11]). The mean number of m-th nearest neighbors of a vertex is

$$z_m = z_1(z_2/z_1)^{m-1} \,, \tag{4}$$

where $z_1 = \langle k \rangle$ is the mean number of the nearest neighbors of a vertex, i.e., the mean degree. $z_2 = \langle k^2 \rangle - \langle k \rangle$ is the mean number of the second nearest neighbors of a vertex. z_2/z_1 is actually the branching coefficient. By using formula (4), one can get ℓ: $z_\ell \sim N$, so $\ell(N) \approx \ln N / \ln(z_2/z_1)$.

Similarly, for the mean number of m-th nearest neighbors of a vertex of degree k, we have

$$z_m(k) = k(z_2/z_1)^{m-1} \,. \tag{5}$$

So, the estimate is $k(z_2/z_1)^{\ell(k)-1} \sim N$ and thus

$$\ell(k) \approx \frac{\ln(N/k)}{\ln(z_2/z_1)} \,. \tag{6}$$

The relation (5) is evident. It also may be obtained strictly by using the Z-transformation technique:

$$z_m(k) = \left[x \frac{d}{dx} \phi_1^k(\phi_1(\ldots\phi_1(x))) \right]_{x=1} \,. \tag{7}$$

$\phi_1(x) = \phi(x)/z_1$ is the Z-transformation of the distribution of the number of edges of an end vertex of an edge with excluded edge itself. $\phi(x)$ is the Z-transformation of the degree distribution of the network: $\phi(x) \equiv \sum_k P(k)x^k$. Formula (7) is a direct consequence of the following features of the configuration model: (i) the network has a locally tree-like structure, (ii) vertices of the network are statistically equivalent, (iii) correlations between degrees of the nearest neighbor vertices are absent. Relation (7) together with $\phi_1(1) = \phi(1) = 1$ readily leads to relation (6).

One point should be emphasized. In the configuration model, the logarithmic size dependence of the mean intervertex distance $\ell(N) \sim \ln N$ is valid only for degree distributions with a finite second moment $\langle k^2 \rangle$. If $\langle k^2 \rangle$ diverges as $N \to \infty$, $\ell(N)$ grows slower than $\ln N$. One can see that the result (6) may be generalized to any given form $\ell(N)$ of the size-dependence of the mean intervertex distance. In this general case, the degree-dependent separation is expressed in terms of the function $\ell(N)$, namely, $\ell(k, N) \sim \ell(N/k)$.

IV. DERIVATIONS

In this section we obtain a degree-dependent intervertex separation for the deterministic graphs of Fig. 1.

Graphs (a) – (d) have a discrete spectrum of vertex degrees with a power-law envelope. Graph (e) has a discrete spectrum of vertex degrees with an exponential envelope. We also list some basic characteristics of these graphs. We stress that the main structural characteristics (clustering, degree–degree correlations [28, 29, 30, 31, 32, 33], etc.) of these deterministic networks are quite close to those of their stochastic analogs (see [14]).

(A) Graph (a) in Fig. 1.—This graph was proposed in Ref. [2] and extensively studied in Ref. [14]. The growth starts from a single edge ($t = 0$). At each time step, each edge of the graph transforms into a triangle. Actually, we have a deterministic version of a stochastic growing network with attachment of a new vertex to a randomly chosen edge, see Ref. [34]. The number of vertices of the graph is $N_t = 1+(3^t+1)/2$. ($t = 0, 1, 2, \ldots$ is the number of the generation.) In the large network limit, the mean degree of the graph is $\langle k\rangle \to 4$.

Degrees of the vertices in the graph take values $k(s) = 2^s$, $s = 1, 2, \ldots, t$. The spectrum of degrees has a power-law envelope. This spectrum corresponds to a continuum scale-free spectrum $P(k) \propto k^{-\gamma}$ with exponent $\gamma = 1 + \ln 3/\ln 2 = 2.585\ldots$. Note that this network has numerous triangles, which suggests high clustering. In more detail, by definition, the average clustering coefficient of a vertex of degree k is

$$C(k) = \left\langle \frac{c(k)}{k(k-1)/2} \right\rangle_k = \frac{\langle c(k)\rangle_k}{k(k-1)/2}\,. \tag{8}$$

Here, $c(k)$ is the number of triangles attached to a vertex of degree k, and $\langle\ \rangle_k$ means the averaging over all vertices of degree k. One can see that in this graph (as well as in its stochastic version)

$$C(k) = \frac{2}{k}\,. \tag{9}$$

This gives, for the mean clustering,

$$\overline{C} = \sum_k P(k)C(k) = \frac{4}{5}\,, \tag{10}$$

while the standard clustering coefficient (transitivity), i.e., the density of loops of length 3 in a network,

$$C = \frac{\sum_k P(k)C(k)k(k-1)}{\sum_k P(k)k(k-1)}\,, \tag{11}$$

approaches zero in the infinite network limit, $C = 0$. Note the difference between the finite mean clustering of the network and its zero clustering coefficient.

One can derive an exact analytical expression for the degree-dependent separation by using recursion relations and the Z-transformation technique. However, here we demonstrate a more rapid way which turns out to be useful in many situations:

(i) Find the mean separation values $\ell_t(s)$ for all kinds of vertices in each of several first generations of the deterministic graph [t is the number of generation, and $k = 2^s$, $s = 1, 2, \ldots, t$];

(ii) by using this array of numbers, guess the form of $\ell_t(s)$;

(iii) check this result by computing directly $\ell_t(s)$ for several extra generations of the graph.

There are few computations in stage (i): we have to find only t values of $\ell_t(s)$ in a t generation of a graph. For sufficiently small networks, these values can be found even without a computer. Step (ii) also turns out to be rather easy since we already know the structure of the analytical expressions for a mean intervertex distance in these networks (see Ref. [14]). In this way, we get

$$\ell_t(s) = \frac{1}{2(N_t-1)}[2(2t-s+5)3^{t-2} - 3^{s-1} + 1]\,. \tag{12}$$

This exact result is valid for $t \geq 1$. An asymptotic form of this expression is

$$\ell(k,N) = \frac{4}{9\ln 3}\ln N - \frac{2}{9\ln 2}\ln k - \frac{k^{\gamma-1}}{6N} + \frac{4}{9}\,\frac{\ln 2}{\ln 3} + \frac{10}{9} + \ldots \tag{13}$$

at large k [$k \gg (\ln N)^{1/(\gamma-1)}$, N is the total number of vertices in the graph]. This leads to result (1).

One can see that the minimum value of $\ell(k)$ is $\ell_{\min} = \ell(k = 2^t) \cong 2t/9$, where $t \cong \ln N/\ln 3$. The maximum number of $\ell(k)$ is $\ell_{\max} = \ell(k = 2) \cong 4t/9$. So, we arrive at relation (3): $\ell_{\max} = 2\ell_{\min}$.

(B) Graph (b) in Fig. 1.—This graph was proposed in Ref. [15]. At each time step, each edge of the graph transforms in the following way: each end vertex of the edge gets a new vertex attached [see Fig. 1, graph (b), instant $0 \to$ instant 1]. This graph is very similar to graph (a). In particular, the exponent of its degree distribution is the same, $\gamma = 1 + \ln 3/\ln 2 = 2.585\ldots$. The difference is that the graph is a tree, so the mean degree $\langle k\rangle \to 2$ as $N \to \infty$.

The total number of vertices in the graph is $N_t = 3^t + 1$. The vertices have degrees $k(s) = 2^s$, where $s = 0, 1, 2, \ldots, t$. In the same way as for graph (a), we find the exact expression

$$\ell_t(s) = \frac{1}{2(N_t-1)}[(4t-2s+9)3^{t-1} - 3^s]\,, \tag{14}$$

which is valid starting with $t = 0$. This leads to the asymptotic relation

$$\ell(k,N) = \frac{2}{3\ln 3}\ln N - \frac{1}{3\ln 2}\ln k - \frac{k^{\gamma-1}}{2N} + \frac{3}{2} + \ldots\,, \tag{15}$$

that is, to result (1).

The minimum value of $\ell(k)$ is $\ell_{\min} = \ell(k = 2^t) \cong t/3$, where $t \cong \ln N/\ln 3$. The maximum value is $\ell_{\max} = \ell(k = 1) \cong 2t/3$, i.e., again, we confirm the validity of relation (3).

(C) Graph (c) in Fig. 1.—At each step, (i) a new vertex becomes attached to each end vertex of each edge of this graph and, simultaneously, (ii) a new vertex becomes attached to each vertex of the graph. This produces a growing deterministic scale-free tree with exponent $\gamma = 3$,

which is a deterministic analog of the Barabási-Albert model [35, 36] (for exact solution of the stochastic model, see Refs. [28, 37, 38]).

The number of vertices in the graph is $N_t = 1 + (4^{t+1} - 1)/3$. Their degrees take values $k(s) = 2^s - 1$, $s = 1, 2, 3, ..., t+1$. The resulting formula for the degree-dependent separation is

$$\ell_t(s \geq 2) = \frac{1}{9(N_t - 1)}[2(6t - 3s + 10)4^t - 4^s - 1]\,. \quad (16)$$

Asymptotically, this is

$$\ell(k, N) = \frac{1}{\ln 4}\ln N - \frac{1}{2\ln 2}\ln k - \frac{k^{\gamma-1}}{9N} + \frac{\ln 3}{2\ln 2} + \frac{2}{3} + \ldots \quad (17)$$

for $k^3 \gg N$ (note that the maximum degree of a vertex in this graph is $k_{\max} \sim N^{1/2}$). This leads to expression (1) with $\gamma = 3$, which coincides with result (6) for uncorrelated networks. This is an understandable coincidence. Indeed, correlations between degrees of the nearest neighbor vertices in this deterministic graph, as well as in the Barabási-Albert model are anomalously week. So, the result must be close to that for an uncorrelated network.

The minimum value of $\ell(k)$ in this graph is $\ell_{\min} = \ell(k = 2^{t+1} - 1) \cong t/2$, where $t \sim \ln N/\ln 4$. The maximum value is $\ell_{\max} = \ell(k = 1) \cong t$, and so the relation (3) is fulfilled.

(D) Graph (d) in Fig. 1.—At each step, (i) a pair of new vertices is attached to ends of each edge of the graph plus (ii) two new vertices are attached to each vertex of the graph. This results in the value of the γ exponent greater than 3, $\gamma = 1 + \ln 5/\ln 2 = 3.322\ldots$.

The number of vertices in the graph is $N_t = (3 \cdot 5^t + 1)/2$. Degrees of the vertices are $k(s) = 3 \cdot 2^{s-1} - 2$, $s = 1, 2, 3, ..., t+1$. The exact expression for the degree-dependent separation is

$$\ell_t(s) = \frac{1}{8(N_t - 1)}[(72t - 36s + 71 + 5^{3-s})5^{t-1} + 2\,5^{s-1} - 6]\,. \quad (18)$$

The corresponding asymptotic expression is of the following form:

$$\ell(k, N) = \frac{6\ln N}{5\ln 5} - \frac{3\ln k}{5\ln 2} - \frac{5^{-\ln 3/\ln 2}}{4N}k^{\gamma-1} + 1.232 + \ldots\,, \quad (19)$$

where the contribution $1.232\ldots = (6\ln(2/3))/(5\ln 5) + (3\ln 3)/(5\ln 2) + 7/12$. Again, now with the graph where $\gamma > 3$, we arrive at formula (1).

In this graph, we have $\ell_{\min} = \ell(k = 3 \cdot 2^t - 2) \cong 3t/5$ and $\ell_{\max} = \ell(k = 1) \cong 6t/5$, where $t \cong \ln N/\ln 5$.

(E) Graph (e) in Fig. 1.—At each time step, a new vertex becomes attached to each vertex of the graph. The growth starts with a single vertex ($t = -1$). The total number of vertices in the graph is $N_t = 2^{t+1}$. The degree distribution is exponential. One can check that the number of vertices of degree k at time t is $N_t(k \leq t) = 2^{t+1-k}$, $N_t(k = t+1) = 2$ (t is assumed to be greater than -1).

By using the procedure that was described above, we find the exact expression:

$$\ell_t(k) = \frac{2^t}{2^{t+1} - 1}(2t + 2 - k)\,. \quad (20)$$

This formula shows that the linear dependence on degree is valid for any k. For the large graphs we have

$$\ell(k, N) \cong \frac{\ln N}{\ln 2} - \frac{k}{2}\,, \quad (21)$$

which confirms formula (2).

In this graph, $\ell_{\min} \cong \ln N/(2\ln 2) \cong \ell_{\max}/2$ which coincides with relation (1).

Graph (e) has a close stochastic analog—a tree, where at each step, a new vertex is attached to a randomly chosen vertex. In principle, this is a solvable model. However, for comparison, we present here the result of the simulation of this stochastic network. Figure 2(a) demonstrates that the dependence $\ell(k)$ in the stochastically growing network is a linear function with the same slope $-1/2$ as in deterministic small world (e) in Fig. 1.

We also considered more general deterministic graphs of this type, where n new vertices become attached to each vertex of a network at each time step. The resulting dependence $\ell(k)$ is a linear function but with slope $-1/(n+1)$. Figure 2(b) shows that $\ell(k)$ of the corresponding stochastically growing networks has the same form.

V. DISCUSSION AND SUMMARY

Several points should be emphasized:

(i) One of the aims of this paper is to demonstrate, how one can study characteristics of growing networks by using simple deterministic graphs. We have found exact expressions for a degree-dependent vertex separation $\ell(k)$ for a number of deterministic graphs and have suggested that $\ell(k)$ of the close stochastic analogs of these graphs behaves similarly. This suggestion is based on already known parallels for other characteristics, however, it is only a suggestion. We have checked it in some situations (see Fig. 2), but cannot prove it.

(ii) We used a set of deterministic graphs, which covers a wide range of typical situations in growing networks. In the scale-free graphs that we used the γ exponent varied in the range $(2, \infty)$. We considered clustered graphs and trees. Furthermore, as is natural, these growing graphs have degree–degree correlations. We observed that the dependence $\ell(k)$ of a deterministic graph with weak degree–degree correlations has a form typical for the configuration model, i.e., uncorrelated network. We indicate that the growing networks that are discussed in this paper grow only due to connection of new vertices to existing ones. New connections cannot emerge between already existing vertices. That is, these networks are "citation graphs". This is a serious restriction.

Note, however, that this class of network is a topic of an overwhelming number of the studies of growing networks. We expect that in more general scale-free networks, instead of $\ell(k) \cong A\ln[N/k^{(\gamma-1)/2}]$, one will find $\ell(k) \cong A\ln N - B\ln k$, where A and B are some constants.

(iii) We believe that the correction term in formula (1) can be hardly observed due to its smallness. It is the form of the main contributions in formulas (1) and (2), that is important. We found that these expressions differ from that for the configuration model, formula (6).

(iv) One should indicate that result (2), i.e., a linear dependence $\ell(k)$, was obtained only for growing trees with an exponential degree distribution. In non-tree growing networks with random attachment (at each time step, a new vertex becomes attached to several randomly chosen vertices), we observed a non-linear dependence.

(v) The relative width of the distribution of the intervertex distance in infinite small worlds approaches zero [9, 14]. In other words, vertices of an infinite small world are almost surely mutually equidistant. This circumstance does not allow one to measure $\ell(k)$ in an infinite network with the small-world effect. However, even in very large real-world networks (e.g., in the Internet [30]), the distribution of the intervertex distance is still broad enough. So, in real networks, $\ell(k)$ is a measurable characteristic.

In conclusion, we have found the mean length of the shortest paths between a vertex of degree k and other vertices in a number of networks with power-law and exponential degree distributions. We have obtained these dependences by using a representative set of deterministic graphs. We have checked that these laws are also realized at least in several stochastically growing networks. We have observed the dependences $\ell(k)$ which strongly differ from those for uncorrelated networks. We believe that our results hold for a wide class of random networks. Our results characterize the compactness of a network from the point of view of a vertex with a given number of connections.

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