

Exactly solvable small-world network

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Abstract. – We present an exact description of a crossover between two different regimes of simple small-world networks. Each of the sites chosen with a probability p from n sites of an ordered system defined on a circle is connected to all other sites selected in such a way. Every link is of a unit length. While p changes from 0 to 1, an averaged shortest distance between a pair of sites changes from $\bar{\ell} \sim n$ to $\bar{\ell} = 1$. We find the distribution of the shortest distances $P(\ell)$ and obtain a scaling form of $\bar{\ell}(p, n)$. In spite of the simplicity of the models under consideration, the results are close to those obtained numerically for usual small-world networks.

A sharp interest on small-world networks began in 1998, when a beautiful paper of Watts and Strogatz [1] was published. A number of publications on the topic is rapidly increasing [2–17]. One of the versions of the model proposed by Watts and Strogatz looks as the following (see ref. [8]). Let n sites be arranged on a circle. Each of several neighbours are connected by unit length links. Then each of those links is rewired with a probability p to one of other randomly chosen sites. Thus, one studies a crossover from an ordered structure at $p = 0$ to a random graph [18]. A lot of real systems are in the state of such a crossover, several examples are listed in Watts’s book [19].

The main quantity which is considered for these models is the shortest distance between two sites of a network (that is a quantity which is much more difficult to calculate than connectivity, for instance). When $p = 0$, the shortest distance averaged over all pairs of sites $\bar{\ell}$ is equal to $n/4$, and $\bar{\ell} \sim \log n$ if $p = 1$ [18, 20].

The central result obtained by Watts and Strogatz [1] is very simple: even addition of a single shortcut to a large system decreases $\bar{\ell}$ noticeably, so that $\bar{\ell}$ is a quantity extremely sensitive to such “global” changes in a network as addition of a shortcut between far sites.

The main problem of the small-network theory is to describe the crossover between the $p = 0$ and $p = 1$ regimes of a finite-size network and to find a scaling form [21] for $\bar{\ell}(p, n)$. One should emphasize that in spite of a lot of effort to understand properties of small-world

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networks, the only exactly proved statement for $\bar{\ell}$ was the following: the scaling variable of the crossover under consideration is pn —see, *e.g.*, [8]. The reason of the failure is the difficulty of calculation of $\bar{\ell}$. In the present communication we consider very simple models with such a crossover between states with two different values of $\bar{\ell}$ for which those calculations may be made exactly. We shall present the scaling form of $\bar{\ell}$ and we shall describe $P(\ell)$ analytically. In fact, we choose toy small-world networks which are very far from the original one [1]. Nevertheless, the answers for $\bar{\ell}$ and $P(\ell)$ are close to the corresponding results obtained numerically for previously studied much more complex small-world networks.

Let us introduce the models. Let n sites be arranged on a circle. Each of them is connected with two nearest neighbours by links of unit length. We shall start from the most simple model with all the bonds on the circle oriented in one direction as shown in fig. 1(a). Afterwards we shall study the model with all undirected links (see fig. 1(b)). We add a new central point to a network and connect it with any other site chosen with the probability p by an undirected link of length equal to $1/2$. In fact, it means that we select some sites with the probability p and then connect all of them by undirected unit length links. Note that such a procedure leads to $\bar{\ell}(p=1) = 1$, unlike the model [1]. Let us find the distribution $P(\ell)$ of the shortest paths between pairs of sites. In the present short communication we use, maybe, the simplest way to find the distribution. At first, we obtain the probability $P(\ell, k)$, *i.e.* the probability that the shortest path from one site to another is ℓ when the distance between them counted along the ring is k : $\sum_{\ell=1}^k P(\ell, k) = 1$. We find $P(\ell \leq k, k)$ for several small values of k and demonstrate that it can easily be described by a general formula for all $\ell \leq k$ and k .

To find $P(\ell, k)$ for the model shown in fig. 1(a), one has to consider all possible configurations of links connecting the center and sites $i = 0, 1, 2, \dots, k$ and to calculate statistical weights for the shortest path to be equal to a particular value ℓ . In such a way, one may easily obtain the following expressions for small ℓ and k :

$$\begin{aligned}
 P(1, 1) &= 1, \\
 P(1, 2) &= p^2, \\
 P(2, 2) &= 1 - p^2, \\
 P(1, 3) &= p^2, \\
 P(2, 3) &= 2p^2(1 - p), \\
 P(3, 3) &= 1 - p^2 \cdot 1 - 2p^2(1 - p), \\
 P(1, 4) &= 1 \cdot p^2(1 - p)^0, \\
 P(2, 4) &= 2p^2(1 - p)^1, \\
 P(3, 4) &= 3p^2(1 - p)^2, \\
 P(4, 4) &= 1 - p^2[1 \cdot (1 - p)^0 + 2(1 - p)^1 + 3(1 - p)^2], \\
 &\dots\dots
 \end{aligned} \tag{1}$$

The subsequent expressions show the same trend. Thus, the general form of $P(\ell, k)$ is

$$\begin{aligned}
 P(\ell < k, k) &= \ell p^2 (1 - p)^{\ell-1}, \\
 P(\ell = k, k) &= 1 - p^2 \sum_{i=0}^{k-1} i (1 - p)^{i-1},
 \end{aligned} \tag{2}$$

and $P(\ell < k, k)$ is independent of k . The distribution of shortest distances between pairs of

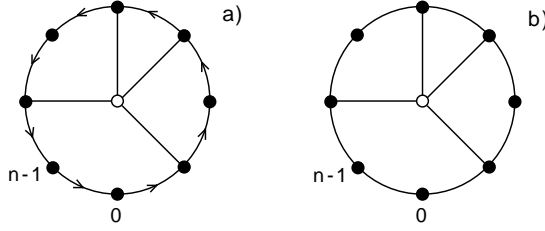


Fig. 1 – Models under consideration. n sites on the circle are connected by unit length links. A central site is connected to a particular site chosen with the probability p by a link of half-length. The links connected with the center are not directed. a) All links on the circle have a defined orientation. b) There are no directed links in the network.

sites for the model under consideration looks as the following:

$$P(\ell) = \frac{1}{n-1} \sum_{k=1}^{n-1} P(\ell, k) = \frac{1}{n-1} \sum_{k=\ell}^{n-1} P(\ell, k). \quad (3)$$

Inserting eqs. (2) into eq. (3), we get immediately

$$P(\ell) = \frac{1}{n-1} [1 + (\ell-1)p + \ell(n-1-\ell)p^2] (1-p)^{\ell-1}. \quad (4)$$

If $p \rightarrow 0$, $P(\ell) \rightarrow 1/(n-1)$ for any $\ell \leq n-1$. If $p \rightarrow 1$, $P(\ell) \rightarrow \delta_{\ell,1}$, where $\delta_{i,k}$ is the Kronecker symbol.

The average shortest distance $\bar{\ell}$, for the model shown in fig. 1(a), is defined in the following way:

$$\bar{\ell} = \sum_{\ell=1}^{n-1} \ell P(\ell), \quad (5)$$

so using the previous distribution one gets

$$\bar{\ell} = \frac{1}{n-1} \left[\frac{2-p}{p} n - \frac{3}{p^2} + \frac{2}{p} + \frac{(1-p)^n}{p} \left(n - 2 + \frac{3}{p} \right) \right]. \quad (6)$$

Thus, $\bar{\ell}(p \rightarrow 0) \rightarrow n/2$ and $\bar{\ell}(p \rightarrow 1) \rightarrow 1$. The distribution $P(\ell)$ is maximal at the point

$$\ell_{\max} = -\frac{1}{\log(1-p)} + \frac{1}{2p} + \frac{n-1}{2} + \sqrt{\frac{1}{\log^2(1-p)} + \frac{5}{4p^2} + \frac{n-3}{2p} + \frac{(n-1)^2}{4}}. \quad (7)$$

Again, $\ell_{\max}(p \rightarrow 0) \rightarrow n/2$ and $\ell_{\max}(p \rightarrow 1) \rightarrow 1$.

Equations (4) and (6) are our main results for the defined model. To obtain a scaling description of the crossover region, one has to pass to the limits $n \rightarrow \infty$ and $p \rightarrow 0$ while the quantities $\rho \equiv pn$ and $z \equiv \ell/n$ are fixed. In this limit, from eq. (4) one obtains a continuous distribution $Q(z, \rho)$ of z :

$$nP(\ell) \equiv Q(z, \rho) = [1 + \rho z + \rho^2 z(1-z)] e^{-\rho z}, \quad (8)$$

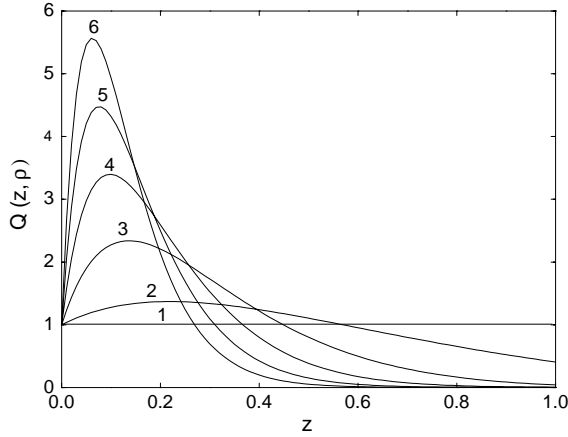


Fig. 2 – The distribution $Q(z, \rho)$ of normalized shortest paths $z \equiv \ell/n$ between pairs of sites for different values of $\rho \equiv pn$ for the model shown in fig. 1(a) (see eq. (8)). Curves labeled by numbers from 1 to 6 correspond to $\rho = 0, 2, 5, 8, 11$ and 14 . For the model without directed links, the corresponding distribution equals $2Q(2z, \rho)$.

$0 \leq z \leq 1$. In fig. 2 we present $Q(z, \rho)$ for several values of ρ as a function of the shortest distance z . The average value of the shortest distance may be found from eq. (6) or (8):

$$\frac{\bar{\ell}}{n} \equiv \bar{z} = \frac{1}{\rho^2} [2\rho - 3 + (\rho + 3) e^{-\rho}]. \quad (9)$$

The corresponding scaling function for ℓ_{\max} is

$$\frac{\ell_{\max}}{n} \equiv z_{\max} = \frac{1}{2\rho} \left[3 + \rho - \sqrt{9 + 2\rho + \rho^2} \right]. \quad (10)$$

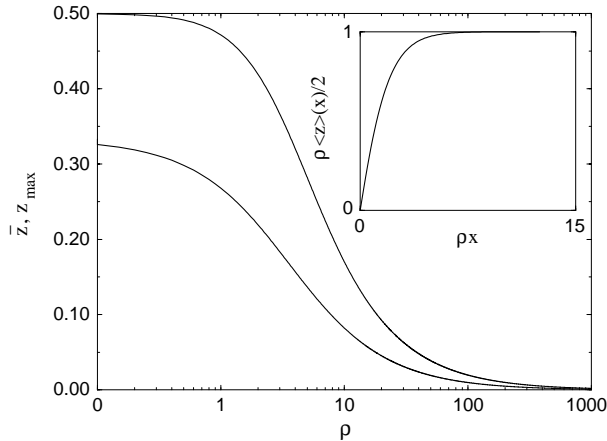


Fig. 3 – The scaling functions $\bar{z} \equiv \bar{\ell}/n$ and $z_{\max} \equiv \ell_{\max}/n$ vs. $\rho \equiv pn$ for the model with directed links. The upper curve corresponds to \bar{z} . Inset: the averaged shortest distance between two sites of the network separated by the “Euclidean” distance $k = xn$ vs. $pk = \rho x$. For the model without directed links, one should change $\bar{z} \rightarrow 2\bar{z}$ and $z_{\max} \rightarrow 2z_{\max}$.

The asymptotic behaviour of \bar{z} is $1/2 - \rho^2/24$ ($\rho \rightarrow 0$) and $2/\rho$ ($\rho \gg 1$) and of z_{\max} is $1/3 - 2\rho/27$ ($\rho \rightarrow 0$) and $1/\rho$ ($\rho \gg 1$). From eqs. (8) and (9), one sees that $(z - \bar{z})^2 \simeq 2/\rho^2$ when $\rho \gg 1$, so $\sqrt{(z - \bar{z})^2/\bar{z}} \rightarrow 1/\sqrt{2}$ in that limit.

In fig. 3 we plot $\bar{\ell}$ and ℓ_{\max} *vs.* ρ . Note that the plot of $\bar{\ell}/n$ resembles the results obtained numerically (*e.g.*, compare with [1, 8, 9]) and by series expansions [12] for ordinary small-world networks, although we consider much simpler network. The analytical result for the distribution $Q(z)$ presented in fig. 2 looks qualitatively similar to the one obtained numerically by Barrat and Weigt [9] for a standard small-world network, although a quantitative analysis cannot be made for these different models.

Using eq. (10) one may obtain also the average shortest distance $\langle \ell \rangle(k)$ between two sites of the network separated by the ‘‘Euclidean’’ distance k . In the continuum limit we get

$$\frac{\langle \ell \rangle(k)}{n} \equiv \langle z \rangle(x) = \frac{2}{\rho} \left[1 - \left(1 + \frac{\rho x}{2} \right) e^{-\rho x} \right] \quad (11)$$

(see inset of fig. 3). This dependence is again similar to the corresponding one for model [1] (compare with [14]).

The model we have studied above looks in some sense rather artificial because of the directed links on the circle. Now we shall consider a more naturally looking model without directed links (see fig. 1(b)). We can proceed in the same way as before to obtain the following probabilities for small k (although the procedure is more tedious in this case, since now to obtain $P(\ell, k)$ one has to count over all configurations of links connecting the center and sites $-k+2, -k+3, \dots, 2k-3, 2k-2$),

$$\begin{aligned} P(1, 1) &= 1, \\ P(1, 2) &= p^2, \\ P(2, 2) &= 1 - p^2, \\ P(1, 3) &= p^2, \\ P(2, 3) &= p^2(2-p)(2-2p), \\ P(3, 3) &= (1-p)^2[1+2p-2p^2], \\ P(1, 4) &= p^2, \\ P(2, 4) &= p^2(2-p)(2-2p), \\ P(3, 4) &= p^2(1-p)^2(2-p)(4-3p), \\ P(4, 4) &= (1-p)^4(1+4p-3p^2), \\ &\dots\dots \end{aligned} \quad (12)$$

(One may check that eqs. (12) hold for any possible n .) As before, these relations may be written in the following general form:

$$\begin{aligned} P(1, 1) &= 1, \\ P(\ell = 1, k) &= p^2, \\ P(2 \leq \ell < k, k) &= p^2(1-p)^{2\ell-4}(2-p)(2\ell-2-\ell p), \\ P(\ell = k, k) &= (1-p)^{2k-4}[1 + (2k-4)p - (k-1)p^2] = 1 - \sum_{\ell=1}^{k-1} P(\ell, k). \end{aligned} \quad (13)$$

As for the previous model, $P(\ell < k, k)$ is independent of k . For an odd n , for instance, one can write

$$P(\ell) = \frac{2}{n-1} \sum_{k=1}^{(n-1)/2} P(\ell, k). \quad (14)$$

Substituting eq. (13) into eq. (14) one gets

$$P(1) = \frac{2}{n-1} \left(1 + \frac{n-3}{2} p^2 \right) \quad (15)$$

and

$$P(\ell \geq 2) = [2 + 4(\ell - 2)p + 2(\ell - 1)(2n - 4\ell - 3)p^2 - 2(2\ell - 1)(n - 2\ell - 1)p^3 + \ell(n - 2\ell - 1)p^4] \frac{(1-p)^{2\ell-4}}{n-1}. \quad (16)$$

This expression for $P(\ell)$ looks different from eq. (4) for the previous model. But when we take the scaling limits, we find the continuum distribution $nP(\ell) \equiv Q'(z, \rho) = 2Q(2z, \rho)$, which differs from the corresponding distribution $Q(z, \rho)$ for the model with oriented links (see eq. (8)) only by a scaling factor of two: $z \rightarrow 2z$ (now $0 \leq z \leq 1/2$). Indeed, one sees that for the model with directed links $\bar{\ell}(p=0) = 1/2$ while for the other model $\bar{\ell}(p=0) = 1/4$. Therefore, after that trivial change of a scale, eqs. (9), (11), and (10) and plots in figs. 2 and 3 are also valid in the present case. In fact, the results for both models under consideration turn to be the same.

There exists another characteristic for random networks —the so-called clustering coefficient $C(p)$, *i.e.* the number of links between neighbours of a site divided by the maximum possible number of links between them and afterwards averaged over all sites. In our case, the coordination number of any site is 2 when $p = 0$, and $C(p=0) = 0$. Models [1] use ordered structures with higher coordination numbers for which $C(p=0) > 0$. Hence, direct comparison of these quantities is impossible (we will present the results for the clustering coefficient in a future publication).

In summary, we have studied two networks which are simple analogies of small-world networks. We have obtained explicit expressions for the distribution $P(\ell)$ of the shortest paths between pairs of sites and for the average shortest path $\bar{\ell}$. The scaling functions look similar to the corresponding ones obtained numerically for more complex small-world networks [1, 8, 9], although, in our case $\bar{\ell} \sim 1/\rho$ instead of $\bar{\ell} \sim \log \rho/\rho$ for the model [1, 12]. The studied networks (especially, the second one) show an ordinary situation in our world in which far connections usually occur through some common center. Presently we are studying some generalizations of the considered models, in particular we include multiple centers in order to make them more realistic.

Note that we have used here the simplest way to find an exact solution of the models under consideration. In fact, we have demonstrated above how to guess the solution, so the problem of searching for explicit solutions for small-world networks remains open.

One should notice, finally, that the above-considered models differ from the original small-world networks [1, 8] in two aspects. One of them is evident: the studied models are of mean-field nature since all the selected sites are connected together by shortcuts (one may also found mean-field approach to the S-W problem in the recent paper [22]). Another aspect is that, although, the original small-world networks are in an ordered state for $p = 0$ and they are random graphs for $p = 1$, in our case, the networks are ordered for both $p = 0, 1$. In our

case, $\bar{\ell}(p = 1) = 1$, and in the model [1] $\bar{\ell}(p = 1) \sim \log n$. Thus, the considered networks are very far from the networks [1, 8]. Nevertheless, as we have shown, the results for our simple models are close to the corresponding answers for the original small-world networks. What is the reason for such a similarity? First of all, we have to say that the obtained crossover behaviour seems to be independent of the mean-field nature of our models. Second, the scaling regime is observed in the limit $p \rightarrow 0$ when all such networks are in the disordered state. Nevertheless, we have solved analytically the model in which the links connected with the central site in fig. 1 are directed chaotically [23], so the crossover is between ordered and disordered states. The obtained results for $\bar{\ell}$ again look similar to fig. 3. Therefore, one may expect that all small-world networks show similar behaviour of $\bar{\ell}$ and $P(\ell, p)$ apart from the $\log(pn)$ term.

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