Three-dimensional Packing Problem: approximation algorithms and performance analysis

F. K. Miyazawa[†] and Y. Wakabayashi[†]

Abstract

This is a short note on approximation results for the Three-dimensional Packing Problem. This problem consists in packing a list of rectangular boxes $L = (b_1, b_2, \ldots, b_n)$ into a rectangular box $B = (l, w, \infty)$, orthogonally, in such a way that the height of the packing is minimized. We consider here two versions of this problem: the oriented and the z-oriented version. In the oriented version, called TPP, the boxes are required to be packed into B orthogonally and oriented in all three dimensions; in the z-oriented version, called TPP^z, the boxes in L are allowed to be rotated around the z-axis.

We mention approximation algorithms for TPP with asymptotic performance bound 3.25, 2.89 and 2.67. We also mention results for TPP^z and for particular instances of both problems. Relations between TPP and TPP^z are also discussed.

1 Introduction

We present here some of the main results on approximation algorithms for the Three-dimensional Packing Problem. We mention only briefly the main ideas behind these algorithms and give their asymptotic performance bounds.

Approximation algorithms for packing problems have been studied since the beginning of seventies, with emphasis on the one- and two-dimensional versions. For a good survey on these two versions the reader may refer to [2]. The three-dimensional case has been less investigated. The first approximation algorithm for the versions we consider here appeared only in 1990 [4, 5].

2 Notation and performance mesures

Let $L = (b_1, b_2, \ldots, b_n)$ be a list of rectangular boxes $b_i = (x_i, y_i, z_i)$, where x_i, y_i and z_i is the **length**, **width** and **height** of b_i, respectively. The **Oriented Three-dimensional Packing Problem**, **TPP**, can be defined as follows. Given a box $B = (l, w, \infty)$ and a list of boxes $L = (b_1, b_2, \ldots, b_n)$, find an orthogonal oriented packing of L into B that minimizes the total height. In the **z-Oriented Three-dimensional Packing Problem**, **TPP^z**, the boxes in L may be rotated around the z-axis (but may not be turned down).

As one can reduce the one-dimensional packing problem to TPP, it follows that both TPP and TPP^z are $\mathcal{NP}$-hard [3].

[†]Instituto de Matemática e Estatística — Universidade de São Paulo — Rua do Matão 1010 — Cidade Universitária — São Paulo, SP — Brazil (e-mail: {keidi,yw}@ime.usp.br).

A very common approach used to attack combinatorial optimization problems that are $\mathcal{NP}$-hard is to develop algorithms with polynomial time complexity that generate solutions close to the optimum ones [3]. To analyse the performance of such algorithms, specially in the case of packing problems, the following performance measures are used.

If $\mathcal{A}$ is an algorithm for TPP or TPP^z and L is a list of boxes, then $\mathcal{A}(\mathbf{L})$ denotes the height of the packing generated by algorithm $\mathcal{A}$ when applied to a list L; and $\mathbf{OPT}(\mathbf{L})$ denotes the height of an optimal packing of L. We say that α is *an* **asymptotic performance bound** of an algorithm $\mathcal{A}$ if there exists a constant β such that for all lists L, in which all boxes have height at most Z, the following holds: $\mathcal{A}(L) \leq \alpha \cdot \text{OPT}(L) + \beta \cdot Z$. Furthermore, if for any small ϵ and any large M, both positive, there is an instance L such that $\mathcal{A}(L) > (\alpha - \epsilon)\text{OPT}(L)$ and $\text{OPT}(L) > M$, then we say that α is *the* **asymptotic performance bound** of algorithm $\mathcal{A}$. We say that α is *an* **absolute performance bound** of an algorithm $\mathcal{A}$ if for all lists L, the following holds: $\mathcal{A}(L) \leq \alpha \cdot \text{OPT}(L)$. If for all $M > 1$, there is an instance L such that $\mathcal{A}(L) > M \cdot OPT(L)$, then we say that $\mathcal{A}$ has an **unbounded worst case performance**.

Given a list of boxes $L = (b_1, \ldots, b_n)$ to be packed into a box $B = (l, w, \infty)$, we assume that each box b_i is of the form $b_i = (x_i, y_i, z_i)$, with $x_i \leq l$ and $y_i \leq w$ or $x_i \leq w$ and $y_i \leq l$. Given a triplet $t = (a, b, c)$, we also refer to each of its elements a, b and c as $x(t)$, $y(t)$ and $z(t)$, respectively.

For each box $b_i = (x_i, y_i, z_i)$, denote by $\rho(b_i)$ the box consisting of the triplet (y_i, x_i, z_i) and let $\mathbf{\Gamma}(\mathbf{L}) = \{(c_1, c_2, \ldots, c_n) : c_i \in \{b_i, \rho(b_i)\}\}$.

Note that, by using a three-dimensional coordinate system, the box $B = (l, w, \infty)$ can be seen as the region $[0, l) \times [0, w) \times [0, \infty)$, and we may define an z**-oriented packing** $\mathcal{P}$ of a list of boxes L into B as a mapping $\mathcal{P} : L' = (b_1, \ldots, b_n) \to [0, l) \times [0, w) \times [0, \infty)$, such that

$$L' \in \Gamma(L), \qquad \mathcal{P}^x(b_i) + x_i \leq l \;\text{ and }\; \mathcal{P}^y(b_i) + y_i \leq w \ ,$$

where $\mathcal{P}(b_i) = (\mathcal{P}^x(b_i), \mathcal{P}^y(b_i), \mathcal{P}^z(b_i))$, $i = 1, \ldots, n$.

And furthermore, if $\mathcal{R}(b_i)$ is defined as

$$\mathcal{R}(b_i) = [\mathcal{P}^x(b_i), \mathcal{P}^x(b_i) + x_i) \times [\mathcal{P}^y(b_i), \mathcal{P}^y(b_i) + y_i) \times [\mathcal{P}^z(b_i), \mathcal{P}^z(b_i) + z_i),$$

then the following must hold

$$\mathcal{R}(b_i) \cap \mathcal{R}(b_j) = \emptyset \qquad \forall i, j,\ 1 \leq i \neq j \leq n \ .$$

If in the above definition we replace $L' \in \Gamma(L)$ by $L' = L$ then we have the concept of **oriented packing** (note that the condition $L' = L$ means that the boxes in L may not be rotated around the z-axis).

In what follows, we may use the term *packing* to refer to both the z-oriented or to the oriented packing. When this may cause a confusion we specify which packing we are referring to.

Given a packing $\mathcal{P}$ of L, we denote by $\mathbf{H}(\mathcal{P})$ the **height** of the packing $\mathcal{P}$, i.e., $H(\mathcal{P}) := \max\{\mathcal{P}^z(b) + z(b) : \ b \in L\}$.

If $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_v$ are packings of disjoint lists $L_1, L_2, \ldots, L_v$, respectively, we define the **concatenation** of these packings as a packing $\mathcal{P} = \mathcal{P}_1 \| \mathcal{P}_2 \| \ldots \| \mathcal{P}_v$ of $L = L_1 \cup L_2 \cup \ldots \cup L_v$, where $\mathcal{P}(b) = (\mathcal{P}_i^x(b), \mathcal{P}_i^y(b), \sum_{j=1}^{i-1} H(\mathcal{P}_j) + \mathcal{P}_i^z(b))$, for all $b \in L_i,\ 1 \leq i \leq v$.

In all algorithms mentioned here the input box B is assumed to be of the form $B = (l, w, \infty)$. Throughout this paper we consider $\mathbf{Z}$ as an upper bound on the height of the boxes in L.

3 Oriented Three-dimensional Packing Problem

In 1990, Li and Cheng [4] described several algorithms for TPP. They showed that generalizations of the Next Fit Decreasing Height (NFDH) and First Fit Decreasing Height (FFDH) strategy, much used in the one-dimensional and two-dimensional packing, does not give good results in the three-dimensional case. More precisely, they showed that, without further refinements, these strategies may lead to unbounded worst case performance. They also described specific algorithms for this problem; in particular, an algorithm whose asymptotic performance bound is 3.25, denoted here by LC. The strategy used in this algorithm is to divide the input list into sublists and apply appropriate algorithms for each sublist, returning a packing that is a concatenation of these individual packings.

Let us give an idea of this algorithm. For that, consider the following *types* (or sets) of boxes:

$$\begin{aligned} T_1 &= \{b:\ x(b) \leq \tfrac{1}{2}l,\ y(b) \leq \tfrac{1}{2}w\}, & T_2 &= \{b:\ x(b) \leq \tfrac{1}{2}l,\ y(b) > \tfrac{1}{2}w\}, \\ T_3 &= \{b:\ x(b) > \tfrac{1}{2}l,\ y(b) \leq \tfrac{1}{2}w\}, & T_4 &= \{b:\ x(b) > \tfrac{1}{2}l,\ y(b) > \tfrac{1}{2}w\}. \end{aligned}$$

The sublists considered in the algorithm LC are either one of these types or subdivisions of them, as indicated in Figure 1 (for ease of notation, all figures are drawn considering $l = w = 1$).

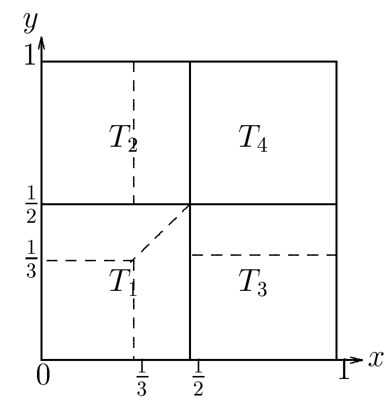

Figure 1: Subdivision of T_1, T_2 and T_3.

The algorithm LC generates an optimum packing $\mathcal{P}'$ of the list of boxes, say L', of type T_4. Since the boxes of type T_4 cannot be packed side by side, they must be packed one on top of the other. Thus, the following inequalities hold:

$$\begin{aligned} H(\mathcal{P}') &\leq \mathrm{OPT}(L), \\ H(\mathcal{P}') &\leq \frac{Vol(L')}{\frac{1}{4}lw} + C' \cdot Z \ \leq\ 4 \cdot \mathrm{OPT}(L') + C' \cdot Z, \end{aligned}$$

where C' is a positive number and $Vol(L)$ is the volume of the boxes in L. In this case, we say that $\mathcal{P}'$ has an **area guarantee** of $\frac{1}{4}$. Of course, if we can improve the area guarantee, then we can have better inequalities for the corresponding packing. We should note that an improvement on the area guarantee of the individual packings may lead to a better asymptotic performance bound of the algorithm.

Li and Cheng show that one can obtain packings for the boxes of type T_1, T_2 and T_3 with area guarantee $\frac{1}{3}$. For that, they consider the sublists indicated in Figure 1 and apply NFDH (Next Fit Decreasing Height) algorithms to generate packings for each of these sublists (see [4]). They show that the performance bound α of the algorithm LC satisfies the following inequality:

$$\alpha \leq \frac{H(\mathcal{P}_4) + H(\mathcal{P}_1 \| \mathcal{P}_2 \| \mathcal{P}_3)}{\max\{H(\mathcal{P}_4), \frac{1}{4}H(\mathcal{P}_4) + \frac{1}{3}H(\mathcal{P}_1 \| \mathcal{P}_2 \| \mathcal{P}_3)}\,,$$

where $\mathcal{P}_i$ is the packing of the boxes in T_i.

Analysing the two possible cases in the denominator of the above fraction, we can conclude that $\alpha \leq 3.25$.

The main drawback of this strategy is that, by using only NFDH algorithms for each of the sublists considered, one cannot improve the area guarantee of the corresponding packing.

In [8] we present an algorithm for TPP whose asymptotic performance bound is less than 2.67. The strategy we use also makes subdivisions of the input list, but generates packings that are obtained by appropriate combinations of different sublists. We propose another subdivison of the boxes in T_1 that gives a packing with area guarantee $\frac{4}{9}$. This motivated us to classify the packings according to their area guarantee: *good*, if it is close to $\frac{4}{9}$; *regular*, if it is close to $\frac{1}{3}$; and *bad*, if it is close to $\frac{1}{4}$.

The algorithm uses a procedure that combines "critical" sublists of boxes of type T_2 or T_3 and generates a good packing of some of these boxes. This procedure also guarantees that all critical boxes that were not packed are either of type T_2 or T_3; furthermore, the non-critical boxes yield a good packing (the right definition of critical boxes is a crucial aspect in this approach). After this process, suppose that the critical boxes of type T_3 have been all packed. Thus, the remaining boxes of type T_3 yield a good packing and all critical boxes are of type T_2. Then, we repeat the same approach again, defining new critical boxes and combining them, reducing the number of critical boxes. This process is repeated a number of times and the non-packed critical sublists are analysed. For some sublists we use the algorithm UD for the bidimensional strip packing problem, developed by Baker, Brown and Kattseff [1], to get inequalities involving the height of the optimum packing.

In 1992, Li and Cheng [6] also presented an on-line algorithm with asymptotic performance bound that can be made as close to 2.89 as desired. Given two numbers r and s, $0 < r, s < 1$, the strategy used by this algorithm is to subdivide the input list L in sublists

$L_{i,j}$ consisting of boxes $b \in L$ with $r^{i+1} < z(b) \leq r^i$ and $s^{i+1} < y(b) \leq s^i$. In this case, each box $b \in L_{i,j}$ is seen as a box with dimensions $(x(b), s^j, r^i)$. The algorithm packs the boxes of $L_{i,j}$ in strips of dimensions $(1, s^j, r^i)$ using a one-dimensional packing algorithm. Finally, the algorithm uses another one-dimensional packing algorithm to pack the strips of height r^i in levels of height r^i. The final packing is the concatenation of all levels generated by the algorithm. The main drawback of this algorithm is that the additive constant β in the asymptotic performance bound is very large.

For the case in which all boxes of L have square bottom, we present [9] an algorithm with asymptotic performance bound less than 2.543. Moreover, for the case in which all boxes have square bottom, we present an algorithm whose asymptotic performance is less than 2.361. Given an integer $m \geq 2$, Miyazawa [7] presented an algorithm with an asymptotic performance bound of $(\frac{m+1}{m})^2$ for boxes in L, such that for all box $b \in L$, $x(b) \leq \frac{l}{m}$ and $y(b) \leq \frac{w}{m}$.

A negative result concerning absolute performance bound for TPP is presented by Li and Cheng in [5]. More precisely, they proved the following result.

Theorem 3.1 *There is no polinomial algorithm for* TPP *with absolute performance bound less than 2, unless $\mathcal{P} = \mathcal{NP}$. This result remains valid even if all boxes have square bottoms and each box $b \in L$ is such that $x(b) \leq \frac{l}{m}$ and $y(b) \leq \frac{w}{m}$.*

4 z-Oriented Three-dimensional Packing Problem

In 1990, Li and Cheng [5] presented TPP^z as a model for the *job scheduling problem in partitionable mesh connected systems*. In this problem a set of jobs $J_1, J_2, \ldots, J_n$ is to be processed in a partitionable mesh connected system that consists of $l \times w$ processing elements connected as a rectangular mesh. Each job J_i is specified by a triplet $J_i = (x_i, y_i, t_i)$ indicating that a submesh of size either (x_i, y_i) or (y_i, x_i) is required by job J_i, and t_i is its processing time. The objective is to assign the jobs to the submeshes so as to minimize the total processing time. The algorithm for TPP^z described in [5] has asymptotic performance bound $4\frac{4}{7}$.

A first idea to solve TPP^z is to adapt algorithms for TPP [4, 6, 8]. A natural approach is to generate for each instance $L = (b_1, b_2, \ldots, b_n)$ a new instance $\phi(L) \in \Gamma(L)$ such that $\phi(L) = (d_1, d_2, \ldots, d_n)$, where

$$d_i = \begin{cases} b_i & \text{if } \ x_i \leq l \ \text{ and } \ y_i \leq w, \\ \rho(b_i) & \text{otherwise} \quad . \end{cases}$$

Then, apply an algorithm for TPP on the list $\phi(L)$.

For each algorithm $\mathcal{A}$ for TPP, let us denote by $\widehat{\mathcal{A}}$ the corresponding algorithm for TPP^z, as described above. That is, for every instance L of TPP^z, algorithm $\widehat{\mathcal{A}}$ applies algorithm $\mathcal{A}$ on the list $\phi(L)$. It is easy to see that the algorithm $\widehat{\mathcal{A}}$ does not preserve the asymptotic performance of the original algorithm $\mathcal{A}$.

The next result shows that there is no algorithm $\widehat{\mathcal{A}}$ for TPP^z, obtained from an algorithm $\mathcal{A}$ for TPP, as described previously, that has an asymptotic performance bound less than 3.

Proposition 4.1 *If $\widehat{\mathcal{A}}$ is an algorithm for* TPP^z *obtained from an algorithm $\mathcal{A}$ for* TPP*, as described above, then the asymptotic performance bound of $\widehat{\mathcal{A}}$ is at least 3.*

Now suppose we have an algorithm $\mathcal{A}$ for TPP^z. It is easy to convert this algorithm for TPP, preserving the same asymptotic performance bound. More precisely, the following holds.

Proposition 4.2 *There is a polynomial reduction of TPP to TPP^z. Moreover, if $\mathcal{A}$ is a polynomial algorithm for TPP^z, such that $\mathcal{A}(L) \leq \alpha \cdot \text{OPT}(L) + \beta \cdot Z$ for every list L in which no box has height greater than Z, then there exists a polynomial algorithm $\mathcal{A}'$ for TPP such that $\mathcal{A}'(L) \leq \alpha \cdot \text{OPT}'(L) + \beta \cdot Z$, where $\text{OPT}'(L)$ is the height of an optimum oriented packing of L.*

In [9] we describe an algorithm for TPP^z that has a asymptotic performance bound less then 2.67. It is based on the algorithm for TPP we mentioned briefly in the previous section [8]. We also describe an algorithm for the special case of TPP^z in which the box B has square bottom and show that its asymptotic performance bound is less than 2.528.

All algorithms mentioned here have complexity $\mathcal{O}(n \log n)$, where n is the number of boxes in the input list L.

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