The Clique Operator, Set Families, and Their Properties

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Abstract

Directed edge path graphs are the intersection graphs of directed paths in a directed tree, viewed as sets of edges. They were introduced by Monma and Wei in 1986, who also gave a polynomial time recognition algorithm. In this work we show that the clique graphs of these graphs are exactly the two-sections of the same kind of path families, and give a polynomial time recognition algorithm for them.

Key words: clique-Helly graphs, computational complexity.

1 Introduction

The clique operator (denoted by K) takes a graph G and returns the intersection graph of the maximal cliques of G. (All graphs are undirected and simple.) Among the best studied graph operators (line graph, block graph, power graph, etc. — see Prisner's book *Graph Dynamics* for a review), K seems to be the most interesting one. For instance, Escalante [3] has shown that for every integer p there is a graph G such that $K^p(G) = G$ and no smaller i satisfies $K^i(G) = G$. Such graphs are called *p-periodic* with respect to K. Also, there are graphs for which the iterated application of K yields bigger and bigger graphs, i. e., they "diverge". No other operator exhibits such richness.

Many questions about K are still open. Perhaps the most important of them is the computational complexity of recognizing clique graphs (graphs that are $K(G)$ for some G). A characterization of clique graphs due to Roberts and Spencer [7] implies that the problem is in NP, but nobody knows whether it is polynomially solvable or NP-complete, or neither.

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A good portion of the research on the clique operator has focused on *graph classes*. Images under K of chordal graphs, clique-Helly graphs, trees, and so on have been characterized. Hamelink [4] has shown that the class of clique-Helly graphs has periodicity 1 (it is *clique-fixed*). In fact, it can be shown that a clique-Helly graph G always has $K^2(G)$ as an induced subgraph, and therefore clique-Helly graphs can have periodicity 1 or 2 only. Notice that if a class is composed solely of graphs of periodicity p, then it is p-periodic, but the converse is not true.

To shed some light on the recognition problem of clique graphs in general, it is relevant to study the computational complexity of the corresponding problem restricted to graph classes. To the best of our knowledge, no class A is known for which $K(A)$ is *not* polynomially recognizable.

In the last few years we have been trying to unify the results about the K operator based on the following:

- Studying graphs together with set families. Although the K operator takes graphs and returns graphs, it can be seen as a two-step process: first construct the family of cliques; then construct the intersection graphs. The intermediate object is a set family. Set families are equivalent to hypergraphs, and they have an important feature that is a great help: duality. Graphs lack duality. There is a notion of duality for planar graphs, but it depends on the plane embedding (it is more a property of the embedding that of the graph itself), is unrelated to clique graphs, and, of course, does not apply to non planar graphs.
- Defining basic operators. Thus, the K operator can be written as $K = LC$, where C is the operator that returns the family of cliques of a graph, and L is the intersection graph operator (takes a family and returns its intersection graph). Two other basic operators are the dual operator D for families (takes a set family and returns another set family where sets and points are interchanged, and the "belongs to" relation is inverted), and the two-section operator S defined as $S = LD$. These operators satisfy $D^2 = I$ (the identity for families) and $SC = I$ (the identity for graphs). Other operators can be defined, such as U that adds singletons to a family, and M that keeps only maximal sets in a family. We build in such a way an "operator algebra" that helps unify and automatize arguments.
- Relating properties to the operators. In the literature a number of important properties appear, such as being a *Helly* family, being a *conformal* family, being a *reduced* family, and being a *separating* family. In fact Helly and conformal are dual concepts, as are reduced and separating. Operator C always returns conformal, reduced families. Operator U separates, while maintaining the image under S. Operator M reduces, while maintining the image under S. Relating these properties to the operators also helps unify and automatize arguments.

As a result of our research we reached the conclusion that mny important results in clique graph theory rely on the fact that $CS = I$ for conformal, reduced families. This includes Hamelink's construction, Roberts and Spencer theorem, and Bandelt and Prisner's partial characterization of clique-fixed classes [1]. It does not include Escalante's results or results related to computational complexity, although it can help on complexity arguments. In fact, we demonstrate the power of our approach proving a new result, namely, the polynomial recognition of the class $K(\mathbf{DE})$, where $\mathbf{DE}$ is the class of intersection graphs of paths of directed trees, viewed as sets of edges [5]. This proof generalizes the corresponding arguments for other related classes whose image under K was shown to be polinomially recognizable recently [6].

Other examples of application of this approach are the study of clique graphs of comparability graphs [2] and clique graphs of planar graphs (Alcon and Gutierrez, in preparation).

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