INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

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S. M. Almeida *C. P. De Mello* *A. Gomide*

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On the representation of a PI-Graph *

Sheila Morais de Almeida† Célia Picinin de Mello† Anamaria Gomide†

Abstract

Consider two parallel lines (denoted by r_1 and r_2). A graph is a *PI graph* (Point-Interval graph) if it is an intersection graph of a family $\mathcal{F}$ of triangles between r_1 and r_2 such that each triangle has an interval with two endpoints on r_1 and a vertex (a point) on r_2. The family $\mathcal{F}$ is the PI representation of G. The PI graphs are an extension of interval and permutation graphs and they form a subclass of trapezoid graphs. In this paper, we characterize the PI graphs in terms of its trapezoid representation. Also we show how to construct a family of trapezoid graphs but not PI graphs from a trapezoid representation of a known graph in this class.

Keywords: Algorithms, interval graphs, permutation graphs, trapezoid graphs, PI graphs.

1 Introduction

We consider simple, undirected, finite graphs $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are the vertex and edge sets, respectively.

A graph is an *intersection graph* if its vertices can be put in a one-to-one correspondence with a family of sets in such way that two vertices are adjacent if and only if the corresponding sets have non-empty intersection.

Consider two parallel lines (denoted by r_1 and r_2). A graph is a *permutation graph* if it is an intersection graph of straight lines (one per vertex) between r_1 and r_2. A graph is a *PI graph* (Point-Interval graph) if it is an intersection graph of triangles between r_1 and r_2 such that each triangle has an interval with two endpoints on r_1 and a vertex (a point) on r_2. The intersection graph of a family of trapezoids that have an interval with two endpoints on r_1 and another one with two endpoints on r_2 is called *trapezoid graph*.

A well known class of intersection graphs is the *interval graphs*, the intersection graph of intervals on a real line.

The PI graphs are an extension of interval and permutation graphs and they form a subclass of trapezoid graphs.

Permutation and interval graphs have been extensively studied since their inception [15, 7, 8, 11] and both have linear-time algorithm for the recognition problem [1, 12, 10].

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†Institute of Computing, University of Campinas (UNICAMP), 13081-970 Campinas, SP, Brazil. sheila@ic.unicamp.br, celia@ic.unicamp.br, anamaria@ic.unicamp.br

Trapezoid graphs class is equivalent to the complements of interval dimension two partial orders. Since Cogis [5], in the early 80s, developed a polynomial time algorithm for the recognition of interval dimension two partial orders, trapezoid graphs recognition may be done in polynomial time too. In [14], Ma presents a trapezoid graph recognition algorithm which runs in time $O(n^2)$. Habib and Möhring [9] and Cheah [3] have also developed polynomial time algorithms for the trapezoid graphs recognition. But PI graphs recognition problem remains still open [2]. This is a motivation to study this class. In Section 2 we characterize the PI graphs in terms of its trapezoid representation and in Section 3, given a graph G that is trapezoid graph but not PI graph, we show how to construct a family of graphs in this class from a trapezoid representation of this known graph.

2 A PI representation

We denote by Π a *trapezoid* between two parallel lines r_1 and r_2 such that Π has one line segment with endpoints on r_1 and another one on r_2 and by Δ a *triangle* between two parallel lines r_1 and r_2 with a line segment on r_1 and a vertex on r_2.

A *trapezoid representation* R of a graph G is a family $\mathcal{F}$ of trapezoids between two parallel lines r_1 and r_2 and G is the intersection graph of $\mathcal{F}$. A *PI representation* R of a graph G is a family $\mathcal{F}$ of triangles between two parallel lines r_1 and r_2 and G is the intersection graph of $\mathcal{F}$. Let G be a graph and $v \in V(G)$. We denote by Π_v the trapezoid of R that corresponds to v and by $\Omega_v^i = [L_v^i, R_v^i]$ the line segment of Π_v that lies on r_i, $i \in \{1, 2\}$. We also denote by $\Omega_u^i << \Omega_v^i$ when $\Omega_u^i \cap \Omega_v^i = \emptyset$ and Ω_u^i lies to the left of the Ω_v^i. The segment Ω_v^2 is denoted by T_v when $L_v^2 = R_v^2$ and thus we have a triangle $\Delta_v = (T_v,\ L_v^1,\ R_v^1)$.

Note that a trapezoid representation of a graph allows triangles (PI graphs are trapezoid graphs). From now on, given R a trapezoid representation of a graph, we consider that any two segments on r_i, $i \in \{1, 2\}$, have distinct endpoints. It is possible, since r_1 and r_2 are real lines.

Let R be a trapezoid representation of a graph G and let Π_u, Π_v and Π_w trapezoids in R such that

$$L_u^2 < R_v^2 < L_w^2 < R_u^2 \text{ and } R_v^1 < L_u^1 < R_u^1 < L_w^1. \tag{1}$$

This triple of trapezoids is called an *obstruction on* r_2 and Π_u is the *center* of the obstruction on r_2. The exchange between r_1 and r_2 gives an obstruction on r_1. We call this triple only by obstruction, when there is no confusion. The triple Π_u, Π_v and Π_w in Figure 1 is an obstruction on r_2. Note that the correspondent vertices v and w of an obstruction are non-adjacent vertices of G. Cheah presents in [3] representations of permutation graphs with similar constructions that he uses to produce a conjecture for the PI graphs recognition.

A graph G has a trapezoid representation R with obstructions on r_2 if, and only if, G has a trapezoid representation R' with obstructions on r_1. In fact, we can exchange r_1 and r_2.

Given R a trapezoid representation of a graph G and $u \in V(G)$ such that Π_u is not the center of any obstruction of R, the next algorithm constructs another trapezoid representation of G from R where the vertex u is represented by a triangle Δ_u.

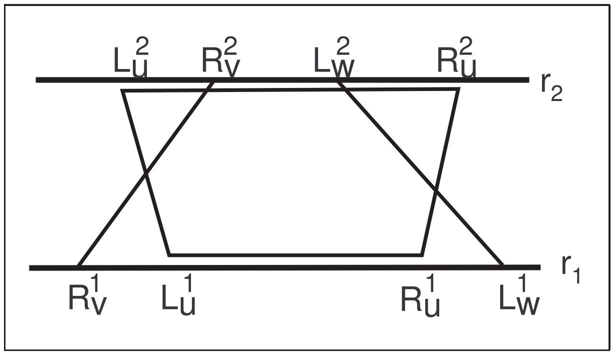

Figure 1: An obstruction on r_2.

Algorithm TRAPtoTRIANG(R, u);
Input: R is a trapezoid representation of a graph G and $u \in V(G)$ such that Π_u is not a center of obstructions of R.
Output: R', a trapezoid representation of a graph G where the vertex u is represented by a triangle Δ_u.

Step 1. If $L_u^2 \neq R_u^2$, then

Step 1.1 $T_l := L_u^2$; $T_r := R_u^2$;
Step 1.2 if there is Π_v such that $\Omega_v^1 << \Omega_u^1$ and $L_u^2 < R_v^2 < R_u^2$,
then $T_r := R_k^2$, where R_k^2 is the leftmost vertex among every R_v^2.
Step 1.3 if there is Π_w such that $\Omega_u^1 << \Omega_w^1$ and $L_u^2 < L_w^2 < R_u^2$,
then $T_l := L_k^2$, where L_k^2 is the rightmost vertex among every L_w^2.
Step 1.4 choose T_u such that $T_l < T_u < T_r$ and do $\Omega_u^2 := T_u$.
Step 1.5 $R := (R \backslash \{\Pi_u\}) \cup \{\Delta_u\}$, where $\Delta_u = (T_u, L_u^1, R_u^1)$.

Step 2. $R' := R$ and return R'.

Lemma 1 *Let R be a trapezoid representation of a graph G and u a vertex of G such that Π_u is not a center of any obstruction of R. Then the Algorithm TRAPtoTRIANG(R,u) transforms Π_u to a triangle Δ_u.*

Proof. If $L_u^2 = R_u^2$, then Π_u is a triangle, only Step 2 is executed, and the lemma follows.

Now, we consider $L_u^2 < R_u^2$ and we suppose that is not possible to choose T_u such that $T_l < T_u < T_r$, in the Step 1.4. Thus, $T_r < T_l$ since all vertices of R are distinct. Since, in Step 1.1, $T_l = L_u^2 < R_u^2 = T_r$, the condition $T_r < T_l$ says that the Step 1.2 or Step 1.3 of the algorithm are executed. If only Step 1.2 (or Step 1.3) is executed, $T_r = R_v^2$, for some Π_v, and $T_l = L_u^2$ ($T_l = L_w^2$, for some Π_w, and $T_r = R_u^2$). In this case, by condition of Step

1.2 (Step 1.3), $T_l = L_u^2 < R_v^2 = T_r$ ($T_l = L_w^2 < R_u^2 = T_r$), a contradiction. Hence both Step 1.2 and Step 1.3 are executed. Therefore, there is Π_v with $\Omega_v^1 << \Omega_u^1$, $L_u^2 < R_v^2 < R_u^2$ and $T_r = R_v^2$ and there is Π_w such that $\Omega_u^1 << \Omega_w^1$, $L_u^2 < L_w^2 < R_u^2$ and $T_l = L_w^2$. Since $T_r < T_l$, the triple Π_v, Π_u and Π_w would be an obstruction of R with Π_u the center of this obstruction, a contradiction. So, $T_l < T_r$ and thus it is possible to choose a vertex T_u such that $T_l < T_u < T_r$ and the Algorithm TRAPtoTRIANG(R,u) makes Π_u into a triangle $\Delta_u = (T_u,\ L_u^1,\ R_u^1)$. ■

Lemma 2 *Let R be a trapezoid representation of a graph G and u a vertex of G such that Π_u is not a center of any obstruction of R. Then the representation obtained by Algorithm TRAPtoTRIANG(R,u) is a trapezoid representation of G.*

Proof. By Lemma 1 the Algorithm TRAPtoTRIANG(R,u) transforms Π_u to a triangle $\Delta_u = (T_u,\ L_u^1,\ R_u^1)$ with $L_u^2 < T_u < R_u^2$. We will show that the TRAPtoTRIANG(R,u) preserves the adjacencies of G.

If condition of Step 1 is not satisfied, then $\Pi_u = \Delta_u$ and only Step 2 is executed, and the Lemma 2 follows. Otherwise, steps 1.1 to 1.5 are executed.

Since these steps of the algorithm only reduces Ω_u^2 to T_u and $T_u \in \Omega_u^2$, no new intersection is created. So, it is sufficient to consider trapezoids of R that have non-empty intersection with Π_u. The algorithm acts only on r_2, then the intersections of the trapezoids with Π_u on r_1 are maintained. Therefore, we can consider only trapezoids Π_v (and Π_w) such that $\Pi_u \cap \Pi_v \neq \emptyset$ ($\Pi_u \cap \Pi_w \neq \emptyset$) and $\Omega_v^1 << \Omega_u^1$ ($\Omega_u^1 << \Omega_w^1$). By Step 1.1, we have $T_l = L_u^2 < R_u^2 = T_r$. If Step 1.2 and Step 1.3 of the Algorithm are not executed, then the vertices R_v^2 and L_w^2 are not between L_u^2 and R_u^2. Since $L_u^2 < T_u < R_u^2$, the adjacencies are preserved.

If Step 1.2 (Step 1.3) of the Algorithm is executed, we have $L_u^2 < R_v^2 < R_u^2$ ($L_u^2 < L_w^2 < R_u^2$). In this case, the algorithm chooses $T_r = R_k^2$ ($T_l = L_{k'}^2$), where R_k^2 ($L_{k'}^2$) is the leftmost (rightmost) vertex on r_2 among every R_v^2 (L_w^2). This implies, by the selection of R_k^2 ($L_{k'}^2$), that in Step 1.4 $L_u^2 < T_u < R_k^2 \leq R_v^2$ ($L_w^2 \leq L_{k'}^2 < T_u < R_u^2$) and the adjacencies are preserved.

If both Step 1.2 and Step 1.3 of the Algorithm are executed, then we have $T_r = R_k^2 \leq R_v^2$ and $L_w^2 \leq L_{k'}^2 = T_l$ where R_k^2 and $L_{k'}^2$ satisfy the condition of these steps. Since, by hypothesis, Π_u is not a center of any obstruction of R, then $L_{k'}^2 < R_k^2$. Hence, by Step 1.4, $L_w^2 \leq L_{k'}^2 = T_l < T_u < T_r = R_k^2 \leq R_v^2$ and, again, the adjacencies are preserved.

So, the new representation obtained by Algorithm TRAPtoTRIANG(R,u) is a trapezoid representation of G. ■

Lemma 3 *Let R be a trapezoid representation of a graph G without obstructions on r_2. The trapezoid representation obtained by Algorithm TRAPtoTRIANG(R,u) does not have obstructions on r_2.*

Proof. Let R' be a trapezoid representation of a graph G obtained from R by Algorithm TRAPtoTRIANG(R,u), where vertex u of G is represented by Δ_u. Suppose by a moment that R' has an obstruction $\mathcal{O}$ generated by the Algorithm TRAPtoTRIANG(R,u). Since the algorithm modifies only Ω_u^2, the triangle Δ_u belongs to $\mathcal{O}$. But Δ_u can not be the center of obstructions of R', since all the vertices of r_2 are distinct.

Suppose that $\mathcal{O} = \{\Pi_v,\ \Delta_u,\ \Pi_w\}$ with Π_v the center of $\mathcal{O}$ and consider $\Omega_u^1 << \Omega_v^1$. (When $\Omega_v^1 << \Omega_u^1$, the proof is analogous.) So, in R', the obstruction satisfies $\Omega_u^1 << \Omega_v^1 << \Omega_w^1$ and $L_v^2 < T_u < L_w^2 < R_v^2$.

Thus, in R', $\Delta_u \cap \Pi_w = \emptyset$ and $\Delta_u \cap \Pi_v \neq \emptyset$. Then, by Lemma 2, $\Pi_u \cap \Pi_w = \emptyset$ and $\Pi_u \cap \Pi_v \neq \emptyset$ in R. So, we have $L_v^2 < R_u^2 < L_w^2 < R_v^2$ in R. Therefore, there was in R an obstruction $\{\Pi_u,\ \Pi_v,\ \Pi_w\}$ with center Π_v, contradicting the fact that R has no obstructions on r_2. ■

Theorem 4 *A graph G is a PI graph if, and only if, G has a trapezoid representation without obstruction on r_2.*

Proof. Let G be a PI graph. Then G has a PI representation R such that each triangle Δ_v has a top vertex T_v, $v \in V(G)$, on r_2. Recall $T_v \neq T_u$ for $v \neq u$. For each T_v, $v \in V(G)$, it is possible to construct a segment $[L_v^2, R_v^2]$ obtaining a trapezoid representation R' of G. To do this, it is sufficient to construct for each two consecutive top vertices T_v and T_u, two disjoint segments $[L_v^2, R_v^2]$ and $[L_u^2, R_u^2]$ such that if $T_v < T_u$ on R, $\Omega_v^2 << \Omega_u^2$ on R'. This is possible because r_2 is a real line. Hence we conclude that R' is a trapezoid representation of G without obstructions on r_2.

Let $R = R_1$ be a trapezoid representation of a graph G without obstructions on r_2. The Algorithm TRAPtoTRIANG(R,u) acts only at trapezoids Π_u that are not centers of obstructions. By Lemma 1, the Algorithm transforms Π_u into Δ_u. By Lemma 2, this new trapezoid representation, R_2, is also a trapezoid representation of G. Since, by hypothesis, R_1 has no obstructions on r_2, then by Lemma 3, R_2 has no obstructions on r_2 too. Then, we use R_2 in the input of the algorithm and so on.

After $|V(G)|$ applications of Algorithm TRAPtoTRIANG(R_i,v) on distinct vertices v of G, we have a PI representation of G. ■

3 The trapezoid graphs that are not PI graphs

In this section we consider graphs that are trapezoid graphs but not PI graphs. We give properties of trapezoid representations of a graph in this class. Recall that from a trapezoid representation of a graph we obtain another one by exchanging r_1 and r_2. Thus, by Theorem 4, a graph G is a trapezoid graph but it is not PI graph if, and only if, every trapezoid representation of G has obstructions on r_1 and on r_2.

Given a trapezoid representation R of a graph such that R has an obstruction, the next theorem exhibits an structure that is necessary not to destroy the obstruction of R.

Theorem 5 *Let R be a trapezoid representation of a graph G and let $\mathcal{O} = \{\Pi_u, \Pi_v, \Pi_w\}$ be an obstruction in R. If at least one of Π_x, Π_y, Π_t and Π_z satisfying*

$$R_v^1 < L_x^1 < R_y^1 < L_u^1 \quad \text{and} \quad R_y^2 < L_u^2 < R_v^2 < L_x^2 \qquad (2)$$

and

$$R_u^1 < L_z^1 < R_t^1 < L_w^1 \quad and \quad R_t^2 < L_w^2 < R_u^2 < L_z^2, \tag{3}$$

does not exist, then it is possible to construct from R a trapezoid representation of a graph G without the obstruction $\mathcal{O}$.

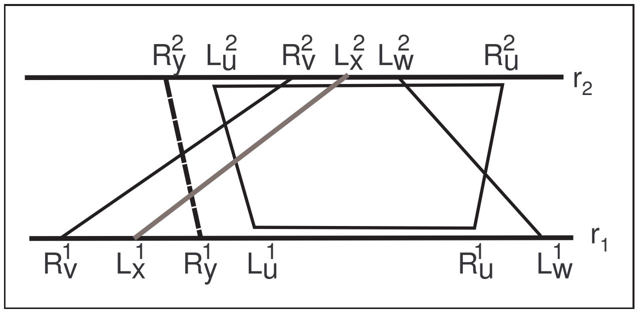

Figure 2: The trapezoids Π_x and Π_y satisfying the condition (2).

Proof. First we consider the trapezoids Π_x, Π_y and the condition (2). (See Figure 2.) The proof for trapezoids Π_z, Π_t and the condition (3) is analogous.

Let $\mathcal{O} = \{\Pi_u, \Pi_v, \Pi_w\}$ be an obstruction of a trapezoid representation R of a graph G with center Π_u. Suppose that there are not trapezoids Π_y such that $R_y^2 < L_u^2 < R_v^2$ and $R_v^1 < R_y^1 < L_u^1$.

Let P be the first endpoint of Ω_p^1 such that $P < R_v^1$. We move the left endpoint of Π_u on r_1 such that the new position of L_u^1 is $P < L_u^1 < R_v^1$ and we call by R' the new trapezoid representation.

Now, we will prove that R' is also a trapezoid representation of G.

The only difference between R and R' is at trapezoid Π_u and on r_1: Ω_u^1 is greater in R' than Ω_u^1 in R but Ω_u^2 was not changed. Hence, if $\Pi \cap \Pi_u \neq \emptyset$ in R, for some trapezoid Π, then $\Pi \cap \Pi_u \neq \emptyset$ in R'.

Now, we shall show that if $\Pi \cap \Pi_u = \emptyset$ in R, for some trapezoid Π, then $\Pi \cap \Pi_u = \emptyset$ in R'. For that, suppose there is a trapezoid Π_k such that $\Pi_k \cap \Pi_u = \emptyset$ in R and $\Pi_k \cap \Pi_u \neq \emptyset$ in R'. Then, $R_v^1 < R_k^1 < L_u^1$ in R. Moreover, since $\Pi_k \cap \Pi_u = \emptyset$ in R, then $R_k^2 < L_u^2$ in R. It follows that Π_k satisfies the condition (2) for trapezoid Π_y in R, a contradiction.

Therefore, R' is a trapezoid representation of G. Moreover, in R', $\Omega_v^1 \cap \Omega_u^1 \neq \emptyset$, so the obstruction $\mathcal{O}$ of R was removed.

Now we suppose that there are not trapezoids Π_x in R such that $R_v^1 < L_x^1 < L_u^1$ and $R_v^2 < L_x^2$.

Let P be the first endpoint of Ω_p^1 such that $L_u^1 < P$. Note that P can be equal to R_u^1. We move the right endpoint of Π_v on r_1 such that the new position of R_v^1 is $L_u^1 < R_v^1 < P$ and we call by R'' the new trapezoid representation.

The only difference between R and R'' is at trapezoid Π_v and on r_1: Ω_v^1 is greater in R'' than Ω_v^1 in R (note that the endpoint L_v^1 and Ω_v^2 were not changed). Hence, if $\Pi \cap \Pi_v \neq \emptyset$ in R, for some trapezoid Π, then $\Pi \cap \Pi_v \neq \emptyset$ in R''.

Suppose that there is a trapezoid Π_k such that $\Pi_k \cap \Pi_v = \emptyset$ in R and $\Pi_k \cap \Pi_v \neq \emptyset$ in R''. Since $\Pi_k \cap \Pi_v \neq \emptyset$ in R'', Π_k has an endpoint on the interval (R_v^1, P). Then, $R_v^1 < L_k^1 < P \leq R_u^1$ in R. By choosing of P, the interval (L_u^1, P) does not have endpoints of trapezoids, then $L_k^1 < L_u^1$. Therefore $R_v^1 < L_k^1 < L_u^1$ in R. Since the intersection of Π_k and Π_v is empty in R, then $R_v^2 < L_k^2$ in R. Hence we conclude that Π_k satisfies the condition (2) for trapezoid Π_x in R, a contradiction.

Since no new intersection was created in R'', it represents the same graph G of R. Moreover, the trapezoid representation R'' has $L_u^2 < R_v^2$ and $L_u^1 < R_v^1$, so the obstruction $\mathcal{O}$ of R was removed. ■

By Theorems 4 and 5, we have the following Collorary.

Corollary 6 *A graph G is a PI graph if and only if there is a trapezoid representation R of G such that for every obstruction on r_2 of R, the condition of Theorem 5 is satisfied.*

Few graphs are known in the class of trapezoid graphs that are not PI graphs [4, 13, 6]. We will show how to construct a family of graphs that belongs to this class from a known graph of the same class.

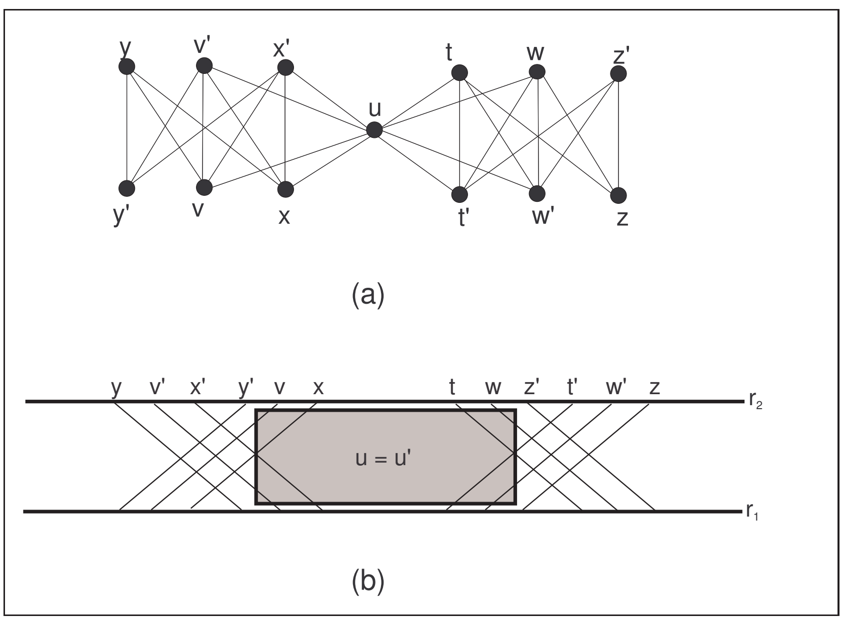

Figure 3: A trapezoid graph G that is not PI graph and a trapezoid representation of G.

Let G be a trapezoid graph that is not PI graph and R a trapezoid representation of G with $\mathcal{O} = \{\Pi_u, \Pi_v, \Pi_w\}$ an obstruction of R on r_2 and $\mathcal{O}' = \{\Pi_{u'}, \Pi_{v'}, \Pi_{w'}\}$ an obstruction of R on r_1. Then R contains trapezoids Π_x, Π_y, $\Pi_{x'}$, $\Pi_{y'}$ satisfying condition (2) and Π_t and Π_z, $\Pi_{t'}$ and $\Pi_{z'}$ satisfying condition (3). (The notation without apostrophe refers to $\mathcal{O}$ and the other one refers to $\mathcal{O}'$.) If $\Pi_u = \Pi_{u'}$, we obtain a representation given by Lin [13]. (See Figure 3.)

Consider the obstruction $\mathcal{O}$ of R. The condition (2) of the Theorem 5 says that $R_v^2 < L_x^2$ and $R_y^2 < L_u^2$. Note that there are no restrictions either on R_x^2 and R_x^1 or on L_y^2 and L_y^1. Thus these vertices can be moved to any position on the right of L_x^2 and of L_x^1 and on the left of R_y^2 and R_y^1, respectively, making new intersections. Similarly, from the condition (3) of the Theorem 5 about R_t^2 and L_z^2, we can move L_t^2 or L_t^1 and R_z^2 or R_z^1 to any position that are less than R_t^2 or R_t^1 and greater than L_z^2 or L_z^1, respectively. The same arguments are valid for an obstruction $\mathcal{O}'$ of R. Therefore, using this liberty for the choice of position of these vertices, we can construct a family of trapezoid graphs that are not PI graphs from a known trapezoid representation of a graph in this class. The Figure 4 shows an element of the family obtained from the trapezoid representation of the Figure 3.

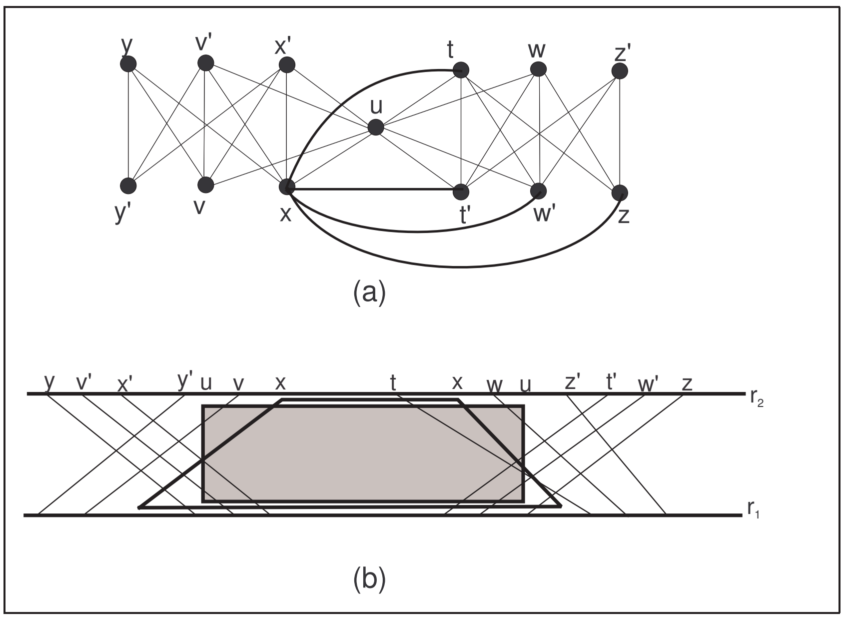

Figure 4: A new trapezoid graph that is not PI graph obtained from the trapezoid representation of the Figure 3.

Let $\Pi \in \{\Pi_x, \Pi_y, \Pi_t, \Pi_z, \Pi'_x, \Pi'_y, \Pi'_t, \Pi'_z\}$. In case Π is equal to some other trapezoid Π' that satisfies the conditions of Theorem 5, then any change at the position of the endpoints of Π must still satisfy the constraints for Π'.

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