The domination and independent domination numbers of Goldberg Graphs

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Abstract

A dominating set of a graph G is a subset $S \subseteq V(G)$ such that every vertex in $V(G)$ either belongs to S or is adjacent to some vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. An independent dominating set of G is a dominating set that is also independent and $i(G)$ is the cardinality of minimum independent dominating set of G. The computational complexity of these problems has led to extensive research focused on establishing bounds or exact values for these parameters in for graph families, especially cubic graphs. Additionally, determining the gap between these two parameters is a challenging problem. In this work, we introduce a family of cubic graphs, called Goldberg Graphs G_l, which generalizes the well-known Goldberg Snarks and show that, for these graphs, $\gamma(G_l) = i(G_l) = \lceil \frac{11l}{5} \rceil$.

1 Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. Throughout this text consider $n = |V(G)|$. Denote the *minimum degree* of G by $\delta(G)$ and the *maximum degree* of G by $\Delta(G)$. If every vertex has degree three, then G is said to be *cubic*. For $u, v \in V(G)$, the *distance* between u and v, $dist(u, v)$, is the number of edges in a shortest path connecting them. The *(open) neighbourhood* of $v \in V(G)$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$; $u \in N(v)$ is a *neighbour* of v. Set $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v.

A set $S \subseteq V(G)$ is a *dominating set* of G if, for every $v \in V(G)$, either $v \in S$ or v is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. If S is a dominating set of G, then S *dominates* G. Moreover, for $v \in V(G)$, we say that v *dominates* all vertices in $N[v]$ and v is *privately dominated* by a subset $S' \subseteq S$ if $N[v] \cap S = S'$.

Domination in graphs is a classical and extensively studied problem in Graph Theory, valued for its rich theoretical structure and its relevance to a wide range of real-world applications. For instance, in wireless sensor networks, a dominating set corresponds to a strategically chosen subset of sensors that guarantees full coverage of the monitored area while minimizing redundancy, thereby reducing energy consumption, communication overhead and maintenance costs [15, 23]. Beyond this, domination and its variants also find applications in distributed systems, resource allocation, robotics, social network analysis, bioinformatics and the study of RNA molecular structures [5, 7, 11, 21, 27].

In 1979, Garey and Johnson [9] have shown that determining $\gamma(G)$ for an arbitrary graph G is an NP-hard problem. Given the computational complexity of the problem, much research in this

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area has focused on establishing bounds on $\gamma(G)$ and determining the parameter for special classes of graphs [2, 12, 13, 29]. In particular, in 1996, Reed [25] proved that every connected graph with a minimum degree $\delta(G) \geq 3$ satisfies $\gamma(G) \leq \frac{3n}{8}$. It is worth noticing that this bound is tight.

Cubic graphs have been a fertile ground for domination research, with many significant findings emerging from investigations into this well-studied class of graphs [14, 18]. Their importance in Graph Theory comes from the fact that numerous graph-theoretic problems can be reduced into their cubic counterparts, making them valuable tools for developing general results and insights [4, 8]. Considering this context, is it not surprising that determining the domination number remains NP-hard when restricted to these graphs [16, 20]. The current best known upper bound on the domination number of connected cubic graphs is $\gamma(G) \leq \left\lfloor \frac{5n}{14} \right\rfloor$, established by Kostochka and Stocker [17] in 2009.

A natural development of the study of domination in graphs is through of generalizations and variants. Indeed, a vast array of variations of the original domination problem have emerged [3, 6, 19] and among these, the concept of an independent dominating set stands out. A set $S \subseteq V(G)$ is *independent* if its elements are pairwise non-adjacent. An *independent dominating set* of G is both dominating and independent. The minimum cardinality of an independent dominating set of G is its *independent domination number* $i(G)$. As its original version, determining $i(G)$ for an arbitrary graph G is an NP-hard problem [9]. Nevertheless, for independent dominating sets, one can also find in the literature many results and practical applications [20, 22].

Note that, by definition, $\gamma(G) \leq i(G)$. However, deciding whether $\gamma(G) = i(G)$ for an arbitrary graph is an NP-complete problem [1]. Even for connected cubic graphs, the difference $i(G) - \gamma(G)$ can be unbounded [28]. In this work, we prove that for the family of Goldberg Graphs G_l, $\gamma(G_l) = i(G_l) = \left\lceil \frac{11l}{5} \right\rceil$.

We introduce Goldberg Graphs as a generalization of the well-known family of Goldberg Snarks. Snarks are connected bridgeless cubic graphs that are not 3-edge-colourable, discovered in the context of the Four-Colour Theorem, occupying a significant position in Graph Theory [26]. In a previous study [24], we determined $\gamma(G)$ and $i(G)$ for certain classes of snarks, showing that $\gamma(G) = i(G)$ for those classes. We further conjectured that this equality holds for every snark. Establishing the equality for Goldberg Snarks provides additional support for this conjecture. However, determining these parameters for Goldberg Snarks proved to be more challenging and required more sophisticated techniques. By generalizing to Goldberg Graphs, we obtain a broader structural framework that preserves the key properties of Goldberg Snarks and enables us to establish our main conclusion.

In order to prove the result, we first demonstrate how to construct an independent dominating set of cardinality $\left\lceil \frac{11l}{5} \right\rceil$ for the Goldberg Graph G_l. Next, we establish some properties of a minimum dominating set of G_l to show that $\left\lceil \frac{11l}{5} \right\rceil$ serves as an upper bound. The key to this proof is Lemma 2, which asserts that $\gamma(G_{l-5}) \leq \gamma(G_l) - 11$.

2 Main results

In 1981, M. K. Goldberg [10] introduced an alternative method for constructing families of snarks. This particular family became known as Generalized Goldberg Snarks. The graphs in this family are built from an odd number of fixed subgraphs. In this work, we further extend this construction by considering graphs that are built from an even number of fixed subgraphs. These graphs, created with this approach, are referred to as *Goldberg Graphs*, and their construction is detailed as follows.

A *basic block* B is a graph isomorphic to the graph shown in Figure 1. Let $\mathcal{G} = \{G_3, G_4, G_5, \ldots\}$ be the family of Goldberg Graphs. Each $G_l \in \mathcal{G}$ is built using $l \geq 3$ copies of block B. We denote by B_i, $0 \leq i < l$, the i-th copy of B in G with $V(B_i) = \{r_i, s_i, t_i, u_i, v_i, w_i, x_i, y_i\}$.

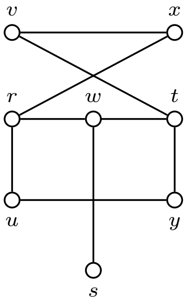


Figure 1: Basic block B used in the construction of Goldberg Graphs.

The first graph in this family, G_3, is obtained by connecting blocks B_i and B_{i+1} by the addition of edges $x_i v_{i+1}$, $y_i u_{i+1}$, and $s_i s_{i+1}$, where $0 \leq i \leq 2$ and indices are taken modulo 3. This graph is illustrated in Figure 2.

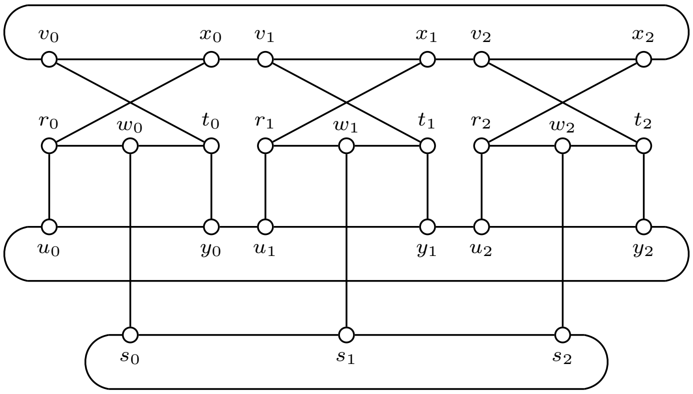


Figure 2: Goldberg Graph G_3.

The remaining graphs in this family, G_l, are obtained recursively: graph G_l, $l \geq 4$, is obtained from graph G_{l-1} and a basic block B_{l-1}, where:

1. $V(G_l) = V(G_{l-1}) \cup V(B_{l-1})$; and
2. $E(G_l) = (E(G_{l-1}) \setminus E^{out}_{l-2}) \cup E^{in}_{l-1} \cup E(B_{l-1})$, with
    - $E_{i,j} = \{x_i v_j, y_i u_j, s_i s_j\}$,
    - $E^{out}_{l-2} = E_{(l-2),0}$ and
    - $E^{in}_{l-1} = E_{(l-2),(l-1)} \cup E_{(l-1),0}$.

Figure 3 illustrates the construction of Goldberg Graph G_l when $l \geq 4$.

We show that, for a Goldberg Graph G_l, $\gamma(G_l) = i(G) = \lceil \frac{11l}{5} \rceil$. Initially, in Theorem 1, we prove the lower bound by constructing an independent dominating set with the desired cardinality.

Theorem 1. *Let G_l be a Goldberg Graph with $l \geq 3$. Then, $\gamma(G_l) \leq i(G) \leq \lceil \frac{11l}{5} \rceil$.*

Proof. We build an independent dominating set S for G, with $|S| \leq \lceil \frac{11l}{5} \rceil$. Let $l = 5t + r$, $t \geq 0$ and $r \in \{0, 1, 2, 3, 4\}$. Let $H_0, H_1, \ldots, H_t$ be subgraphs of G such that: $H_i = G[\cup_{j=0}^{4} V(B_{5i+j})]$

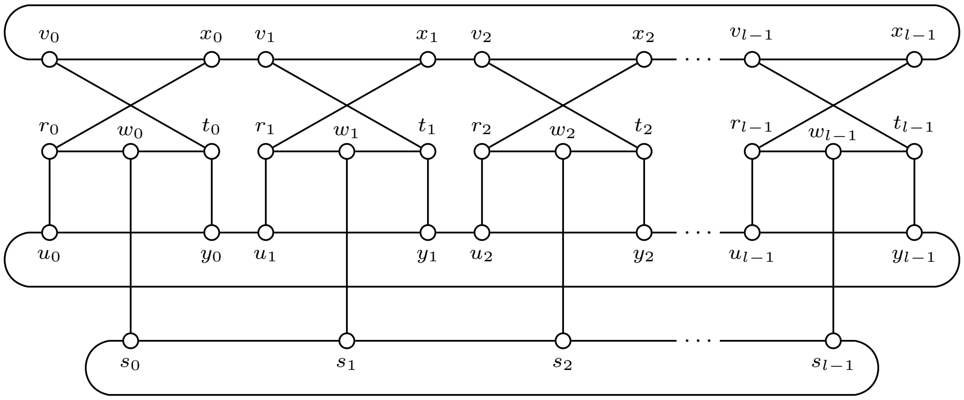
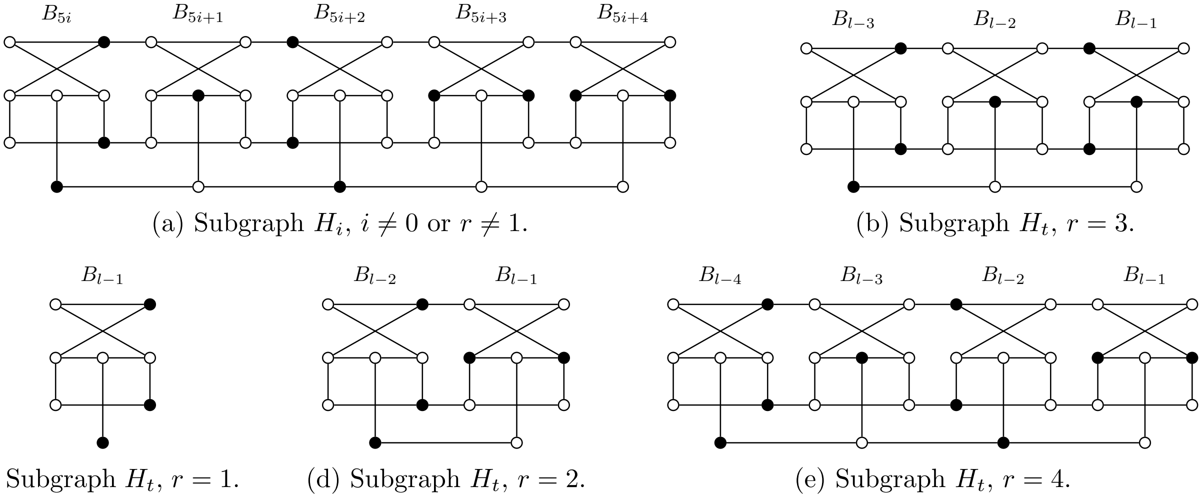


Figure 3: Construction of Goldberg Graph G_l with $l \geq 4$.

if $0 \leq i < t$ and $H_t = G[\cup_{j=0}^{r-1} V(B_{l-r+j})]$ if $r \neq 0$. Note that $|V(H_i)| = 40$ for $0 \leq i < t$ and $|V(H_t)| = 8r$. Figure 4 exhibits subgraphs H_i, when $i \neq 0$ or $r \neq 1$, and H_t. We now construct an independent dominating set S for G where $S = \cup_{i=0}^{t} S_i$ such that, for $0 \leq i < t$,

$$S_i = \begin{cases} \{w_{5i}, x_{5i}, y_{5i}, w_{5i+1}, s_{5i+2}, u_{5i+2}, v_{5i+2}, r_{5i+3}, t_{5i+3}, r_{5i+4}, t_{5i+4}\} & \text{if } i = 0 \text{ and } r = 1; \\ \{s_{5i}, x_{5i}, y_{5i}, w_{5i+1}, s_{5i+2}, u_{5i+2}, v_{5i+2}, r_{5i+3}, t_{5i+3}, r_{5i+4}, t_{5i+4}\} & \text{otherwise; and} \end{cases}$$

$$S_t = \begin{cases} \emptyset & \text{if } r = 0; \\ \{s_{l-1}, x_{l-1}, y_{l-1}\} & \text{if } r = 1; \\ \{s_{l-2}, x_{l-2}, y_{l-2}, r_{l-1}, t_{l-1}\} & \text{if } r = 2; \\ \{s_{l-3}, x_{l-3}, y_{l-3}, w_{l-2}, u_{l-1}, v_{l-1}, w_{l-1}\} & \text{if } r = 3; \\ \{s_{l-4}, x_{l-4}, y_{l-4}, w_{l-3}, s_{l-2}, u_{l-2}, v_{l-2}, r_{l-1}, t_{l-1}\} & \text{if } r = 4. \end{cases}$$



(a) Subgraph H_i, $i \neq 0$ or $r \neq 1$. (b) Subgraph H_t, $r = 3$.

(c) Subgraph H_t, $r = 1$. (d) Subgraph H_t, $r = 2$. (e) Subgraph H_t, $r = 4$.

Figure 4: Subgraphs used in the construction of an independent dominating S set of G_l. The dark vertices belong to S.

An inspection of the figures shows that the set S is indeed an independent dominating set. In the case where $i = 0$ and $r = 1$, the set S_i can be verified similarly to the case shown in the figure. By construction, $|S_i| = 11$ when $0 \leq i < t$ and $|S_t| = 2r + 1$ when $r \neq 0$. Then, $i(G) \leq 11t + 2r + 1$ when $r \neq 0$ and $i(G) \leq 11t$ when $r = 0$. In both cases, $\gamma(G) \leq i(G) \leq \lceil \frac{11l}{5} \rceil$, which concludes the proof. □

In the following lemma, we establish several properties of a minimum dominating set S of G_l, $l \geq 5$. In particular, we explore the properties of $S \cap V(B_i)$. This lemma is used extensively in the remaining of this paper.

Lemma 1. *Let G_l be a Goldberg Graph with $l \geq 5$ and let S be a minimum dominating set of G_l. Let $S_i = S \cap V(B_i)$ such that B_i is a block of G_l, $0 \leq i < l$. Then, the following claims are true.*

1. *Set S_i is not empty.*
2. *If $|S_i| = 1$, then $S_i = \{w_i\}$, $\{x_{i-1}, y_{i-1}\} \subseteq S_{i-1}$ and $\{u_{i+1}, v_{i+1}\} \subseteq S_{i+1}$. Moreover, $|S_{i-1}| \geq 3$, $|S_{i+1}| \geq 3$, $|S_{i-2}| \geq 2$ and $|S_{i+2}| \geq 2$.*
3. *If $|S_i| = 2$, then we have $S_i \in S_i^1 \cup S_i^2$ such that $S_i^1 = \{\{r_i, t_i\}, \{r_i, v_i\}, \{r_i, y_i\}, \{t_i, x_i\}, \{t_i, u_i\}\}$ and $S_i^2 = \{\{r_i, w_i\}, \{s_i, w_i\}, \{t_i, w_i\}, \{u_i, w_i\}, \{v_i, w_i\}, \{w_i, x_i\}, \{w_i, y_i\}\}$. Also, $|S_{i-1}| \geq 2$ and $|S_{i+1}| \geq 2$.*
    - *(a) If $S_i \in S_i^1$, then either $|S_{i-1}| \geq 3$ or $|S_{i+1}| \geq 3$.*
    - *(b) If $S_i \in S_i^2$, then $(|S_{i-3}| + |S_{i-2}| + |S_{i-1}|) \geq 7$ and $(|S_{i+1}| + |S_{i+2}| + |S_{i+3}|) \geq 7$.*
    - *(c) If $|S_{i+j}| = 2$ for every $0 \leq j \leq 3$, then it follows that $S_i = \{r_i, y_i\}$, $S_{i+1} = \{w_{i+1}, x_{i+1}\}$, $S_{i+2} = \{u_{i+2}, w_{i+2}\}$ and $S_{i+3} = \{r_{i+3}, v_{i+3}\}$ or $S_i = \{t_i, x_i\}$, $S_{i+1} = \{w_{i+1}, y_{i+1}\}$, $S_{i+2} = \{v_{i+2}, w_{i+2}\}$ and $S_{i+3} = \{t_{i+3}, u_{i+3}\}$.*

□

From Lemma 1, we obtain the following corollary.

Corollary 1. *Let G_l be a Goldberg Graph with $l \geq 5$. Let S be a minimum dominating set of G_l. Let $S_i = S \cap V(B_i)$ such that B_i is the basic block of G_l and $0 \leq i < l$. Then, considering modular operations on the indices,*

1. *there are no blocks B_i, B_{i+1}, B_{i+2}, B_{i+3}, B_{i+4} such that $|S_j| = 2$, $i \leq j \leq i + 4$. We call this configuration* prohibited*.*
2. *If $|S_i| = 1$ for some i, then $\sum_{j=i-2}^{i+2} |S_i| \geq 11$.* □

Theorem 2 below proves the result for small l.

Theorem 2. *Let G_l be a Goldberg Graph with $l \in \{3, 4, 5, 6, 7\}$. Then, $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$.* □

Next, we find an upper bound on $\gamma(G_{l-5})$ in terms of $\gamma(G_l)$, upon which our main result is based.

Lemma 2. *Let G_l be a Goldberg Graph with $l \geq 8$. Then, $\gamma(G_{l-5}) \leq \gamma(G_l) - 11$.*

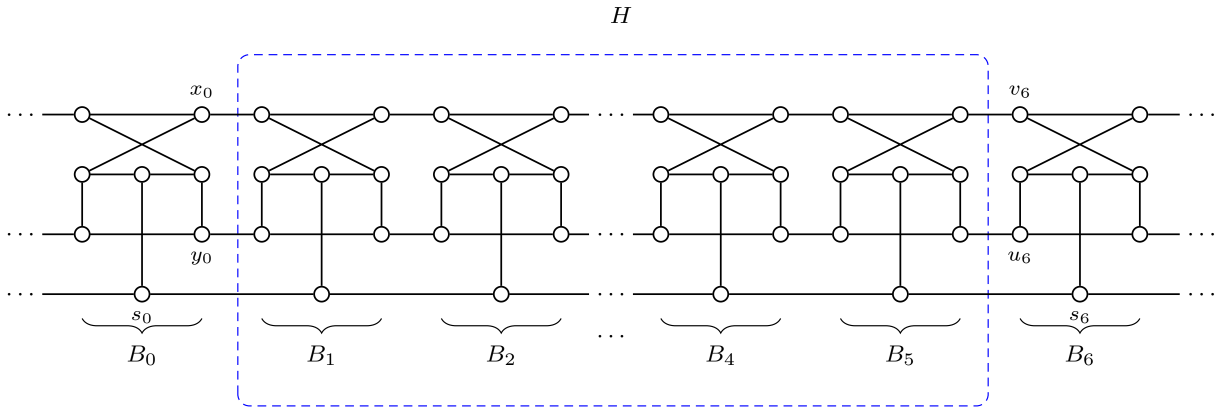


Figure 5: Subgraph H is built with B_i, $1 \le i \le 5$.

Proof. By the recursive construction of the family, we can build G_l from G_{l-5} and five blocks B_i. Let $H = G_l[\cup_{i=1}^{5} V(B_i)]$ and $\overline{H} = G_l[V(G_l) \setminus V(H)]$. Figure 5 sketches the construction of H. Note that $\overline{H}$ is isomorphic to a subgraph of G_{l-5} – the one obtained by removing edges s_0s_6, x_0v_6 and y_0u_6. Let S be a minimum dominating set of G_l. Let $S_H = S \cap V(H)$, $S_{\overline{H}} = S \setminus S_H$ and $S_i = S \cap V(B_i)$.

Initially, we prove $|S_H| \ge 10$. By contradiction, suppose $|S_H| \le 9$. By Lemma 1, $|S_i| \ge 1$ for every $1 \le i \le 5$. We conclude, by counting, that there exists at least one j, $1 \le j \le 5$, such that $|S_j| = 1$.

Suppose $j = 1$. By Lemma 1, $\sum_{i=1}^{3} |S_i| \ge 6$. Therefore, $\sum_{i=4}^{5} |S_i| \le 3$. This implies either $|S_4| = 1$ or $|S_5| = 1$. In both cases, we conclude that, by Lemma 1, $\sum_{i=4}^{5} |S_i| \ge 4$, a contradiction. Thus, $j \ne 1$ and, by symmetry, $j \ne 5$. Suppose $j = 2$. Then, $\sum_{i=1}^{4} |S_i| \ge 9$. Since $|S_5| \ge 1$, $|S_H| > 9$, also a contradiction. Hence, $j \ne 2$ and $j \ne 4$. Suppose $j = 3$. In this case, $\sum_{i=1}^{5} |S_i| \ge 11$. Then, $j \ne 3$ and $|S_H| \ge 10$.

Let $T = \{s_0, x_0, y_0, u_6, v_6, s_6\}$. By construction, $S_{\overline{H}}$ dominates all the vertices in $V(\overline{H}) \setminus T$ and $S_H \cup S_1 \cup S_6$ dominates the vertices of T. If $S_{\overline{H}}$ also dominates T, then $S_{\overline{H}}$ is a dominating set of G_{l-5}. On the other hand, if $S_{\overline{H}}$ does not dominate any vertex of T, then $S_{\overline{H}} \cup \{s_0, x_0, y_0\}$ is a dominating set of G_{l-5}.

In the remaining of the proof, we build a dominating set S' for G_{l-5} using $S_{\overline{H}}$, such that $|S'| \le |S| - 11$. Thus,

$$\gamma(G_{l-5}) \le |S'| \le |S| - 11 = \gamma(G_l) - 11,$$

which concludes the proof. The proof is divided into cases, according to the cardinality of S_H. Most of the conclusions follow either by Lemma 1 or by Corollary 1. In order to not let the proof too much repetitive, we omit the references.

Case 1. $|S_H| = 10$.

In this case, we show that $T \subseteq S_{\overline{H}}$ and, under this hypothesis, there exists S' satisfying the required conditions. In order to see this, suppose $T \subseteq S_{\overline{H}}$. If $x_6 \in S_6$, then $S' = (S_{\overline{H}} \setminus \{u_6, v_6\}) \cup \{y_6\}$ dominates G_{l-5}. Figure 6 illustrates this case. Similarly, if $y_6 \in S_6$, $S' = (S_{\overline{H}} \setminus \{u_6, v_6\}) \cup \{x_6\}$. In both cases,

$$|S'| \le |S_{\overline{H}}| - 1 \le |S| - |S_H| - 1 = |S| - 10 - 1 = |S| - 11. \qquad (1)$$

Now, suppose $x_6, y_6 \notin S_6$. If s_7 is not privately dominated by s_6, i.e. $N[s_7] \cap S \ne \{s_6\}$, we conclude that $S' = (S_{\overline{H}} \setminus \{s_6, u_6, v_6\}) \cup \{r_6, t_6\}$ dominates G_{l-5} and $|S'| \le |S_{\overline{H}}| - 1$ and the

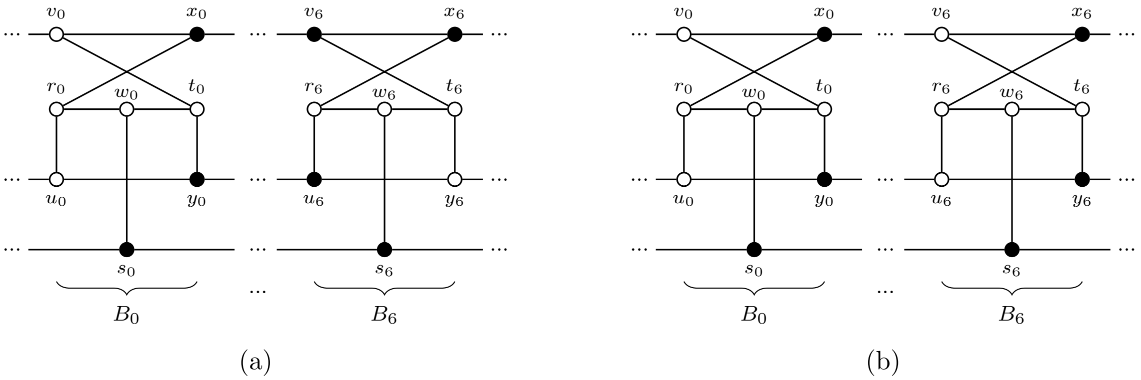


Figure 6: In (a), $x_6 \in S_6$; and in (b), we removed $\{u_6, v_6\}$ and added y_6 to S'.

result follows by (1). Suppose that s_7 is privately dominated by s_6. Since $N[s_7] \cap S = \{s_6\}$, we have $|S_7| \geq 2$. First, suppose $|S_7| \geq 3$. Let $\{a, b, c\}$ be a subset of distinct vertices of S_7. Then, $S' = (S_{\overline{H}} \setminus \{a, b, c, s_6, u_6, v_6\}) \cup \{r_6, t_6, s_7, x_7, y_7\}$ dominates G_{l-5}, as exemplified in Figure 7. Observe that all the vertices in B_6 and B_7 are dominated by S'. Moreover, $\{s_7, x_7, y_7\} \subseteq S_7$, which ensures that the vertices in B_8 remain dominated. Thus, $|S'| \leq |S_{\overline{H}}| - 1$ and the result follows by (1).

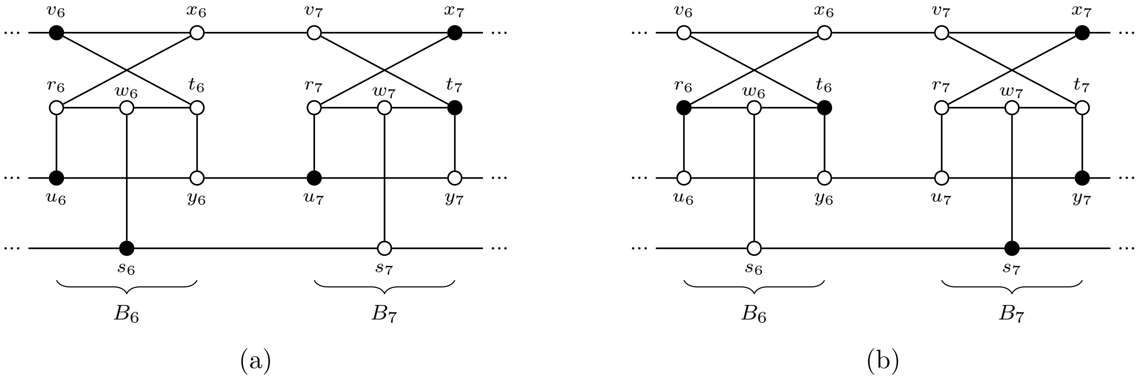


Figure 7: In (a), s_6 privately dominates s_7; and in (b), we removed $\{s_6, u_6, v_6, t_7, u_7\}$ and added $\{r_6, t_6, s_7, y_7\}$ to S'.

In order to conclude the case in which $x_6, y_6 \notin S_6$, it remains to consider $|S_7| = 2$. Since $x_6, y_6 \notin S_6$, set $\{u_7, v_7\}$ must be dominated by S_7. Thus, we conclude that S_7 is equal to one of $\{r_7, t_7\}$, $\{r_7, v_7\}$ or $\{t_7, u_7\}$. Note that, in all these cases, there is no vertex of B_8 dominated by S_7. Then, $S' = (S_{\overline{H}} \setminus (S_7 \cup \{s_6, u_6, v_6\})) \cup \{w_6, s_7, u_7, v_7\}$ is a dominating set of G_{l-5}.

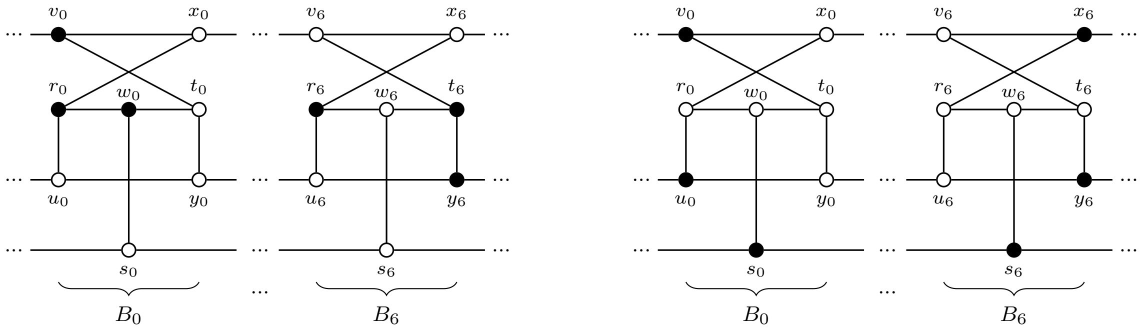
It remains to prove that $T \subseteq S_{\overline{H}}$ or it can be adjusted so as to have this property. Initially, suppose $|S_5| = 1$. Therefore, $\{u_6, v_6\} \subseteq S_6$. However, since some vertex of $N[w_6]$ needs to be in S_6 so as to dominate w_6, we can adjust S_6 so as $s_6 \in S_6$. By symmetry, the same argument can be applied to $\{s_0, x_0, y_0\}$. Hence, if $T \not\subseteq S_{\overline{H}}$, then either $|S_1| \geq 2$ or $|S_5| \geq 2$. Next, we show that this implies that it is not possible to dominate H with only 10 vertices.

Consider $T \not\subseteq S_{\overline{H}}$. Since $|S_{\overline{H}}| = 10$ and there are no prohibited configurations, we conclude that there exists at least one j, $1 \leq j \leq 5$, such that $|S_j| = 1$. Suppose $j = 1$. Then, by hypothesis, since $|S_1| = 1$, $|S_5| \geq 2$. Moreover, $\sum_{i=1}^{3} |S_i| \geq 6$. If $|S_4| = 1$, then $\sum_{i=3}^{5} |S_i| \geq 7$ and $|S_{\overline{H}}| > 10$, a contradiction. Therefore, $|S_4| \geq 2$. We conclude that $|S_4| = |S_5| = 2$. However, when $|S_4| = 2$ we must have either $|S_3| \geq 3$, $|S_5| \geq 3$ or $\sum_{i=1}^{3} |S_i| \geq 7$, which implies $|S_{\overline{H}}| > 10$. Thus, $j \neq 1$ and, by symmetry, $j \neq 5$. Suppose $j = 2$. Then, $|S_1| \geq 3$, $|S_3| \geq 3$ and $|S_4| \geq 2$, that is, $\sum_{i=1}^{4} |S_i| \geq 9$. Therefore, $|S_5| = 1$. However, this implies that $|S_4| \geq 3$ and $\sum_{i=1}^{5} |S_i| > 10$, a contradiction. Thus, $j \neq 2$ and, by symmetry, $j \neq 4$. At last, if $j = 3$, then $\sum_{i=1}^{5} |S_i| \geq 11$, a contradiction again and hence, $T \subseteq S_{\overline{H}}$.

Case 2. $|S_H| = 11$.

If the vertices in T are dominated by $S_{\overline{H}}$, we conclude that $S' = S_{\overline{H}}$, $|S'| \leq |S| - 11$ and the result follows. Furthermore, the result also follows when $\{s_0, x_0, y_0\} \subseteq S_0$ or when $\{s_6, u_6, v_6\} \subseteq S_6$. Suppose that $\{s_0, x_0, y_0\} \not\subseteq S_0$ and $\{s_6, u_6, v_6\} \not\subseteq S_6$. Then, as shown in the previous case, this implies that $|S_1| \geq 2$ and $|S_5| \geq 2$.

Consider, initially, $|S_0| \geq 3$ and $|S_6| \geq 3$. Let $\{a, b, c\}$ and $\{d, e, f\}$ be sets of vertices of S_0 and S_6, respectively. Then, $S' = (S_{\overline{H}} \setminus \{a, b, c, d, e, f\}) \cup \{s_0, u_0, v_0, s_6, x_6, y_6\}$ is a dominating set of G_{l-5}, as illustrated in Figure 8. Moreover, $|S'| \leq |S_{\overline{H}}| \leq |S| - |S_H| = |S| - 11$.



(a) (b)

Figure 8: In (a), $\{r_0, v_0, w_0\} \subseteq S_0$ and $\{r_6, t_6, y_6\} \subseteq S_6$; and in (b), these vertices are replaced by $\{u_0, v_0, s_0\}$ and $\{s_6, x_6, y_6\}$.

Next, suppose $|S_0| \leq 2$ or $|S_6| \leq 2$. Recalling that $|S_1| \geq 2$ and $|S_5| \geq 2$, we show that, in these cases, either the result follows or it is not possible to dominate H with only 11 vertices.

Since $|S_H| = 11$, there is some j, $1 \leq j \leq 5$, such that $|S_j| \neq 2$. First, suppose that there exists at least one j for which $|S_j| = 1$. Since $|S_1| \geq 2$ and $|S_5| \geq 2$, $j \in \{2, 3, 4\}$. Consider $j = 2$. Then, $|S_1| = |S_3| = 3$ and $|S_4| = |S_5| = 2$. Thus, $S_1 = \{s_1, x_1, y_1\}$. Recall that, as shown in the previous case, we can adjust S_1 so as $s_1 \in S_1$. This implies that s_0 is the only vertex in B_0 dominated by S_1, which means that $\{x_0, y_0\}$ is dominated by $S_{\overline{H}}$. Since $\sum_{i=2}^{4} |S_i| < 7$ and $|S_5| = 2$, we conclude that $|S_6| \geq 3$. Let $\{a, b, c\}$ be a set of distinct vertices of S_6. Then, $S' = (S_{\overline{H}} \setminus \{a, b, c\}) \cup \{s_6, x_6, y_6\}$ is a dominating set of G_{l-5}. Moreover, $|S'| \leq |S_{\overline{H}}| \leq |S| - |S_H| = |S| - 11$. By symmetry, the result also follows when $j = 4$.

Now, suppose $j = 3$. Then, $|S_2| = |S_4| = 3$ and $|S_1| = |S_5| = 2$. By hypothesis, $|S_0| \leq 2$ or $|S_6| \leq 2$. Without loss of generality, suppose $|S_0| \leq 2$. However, $|S_0| > 1$ because $|S_0| = 1$ would imply $|S_1| \geq 3$, a contradiction. Therefore, we can assume $|S_0| = 2$. Observe that $\sum_{i=1}^{3} |S_i| < 7$ and $|S_1| = 2$, which implies $|S_{l-1}| \geq 3$. Since $|S_5| = 2$, it follows that $|S_6| \geq 2$. Suppose $|S_6| \geq 3$. Then, $S' = (S_{\overline{H}} \setminus (S_{l-1} \cup S_0 \cup S_6)) \cup \{s_{l-1}, u_{l-1}, v_{l-1}, r_0, t_0, s_6, x_6, y_6\}$ is a dominating set of G_{l-5} and the result follows. Suppose $|S_6| = 2$. Similarly to S_{l-1}, we have that $|S_7| \geq 3$ as $\sum_{i=3}^{5} |S_i| < 7$ and $|S_6| = 2$. Then, $S' = (S_{\overline{H}} \setminus (S_{l-1} \cup S_0 \cup S_6 \cup S_7)) \cup \{s_{l-1}, u_{l-1}, v_{l-1}, r_0, t_0, r_6, t_6, s_7, x_7, y_7\}$ is a dominating set of G_{l-5} and the result follows.

It remains to consider the case in which there is exactly one j, $1 \leq j \leq 5$, such that $|S_j| = 3$. Note that, in this case, the remaining S_i, $i \in \{1,2,3,4,5\} \setminus j$, have $|S_i| = 2$. Initially, consider $j = 1$. Since $|S_2| = |S_3| = |S_4| = |S_5| = 2$, we have either $S_2 = \{r_2, y_2\}$, $S_3 = \{w_3, x_3\}$, $S_4 = \{u_4, w_4\}$ and $S_5 = \{r_5, v_5\}$ or $S_2 = \{t_2, x_2\}$, $S_3 = \{w_3, y_3\}$, $S_4 = \{v_4, w_4\}$ and $S_5 = \{t_5, u_5\}$. Suppose $S_2 = \{r_2, y_2\}$, $S_3 = \{w_3, x_3\}$, $S_4 = \{u_4, w_4\}$ and $S_5 = \{r_5, v_5\}$. Note that this implies $\{s_1, x_1\} \subseteq S_1$ and $\{s_6, u_6\} \in S_6$. Since s_1 and x_1 do not dominate t_1 and y_1, we must have one of $\{t_1, y_1\}$ in S_1. Then, we conclude that, s_0 is the only vertex of B_0 dominated by vertices in S_1. However, no vertex of B_6 is dominated by S_5, and, also, s_0 is dominated by s_6 in S'. We conclude that $S' = S_{\overline{H}}$ is a dominating set of G_{l-5}. For $S_2 = \{t_2, x_2\}$, $S_3 = \{w_3, y_3\}$, $S_4 = \{v_4, w_4\}$ and $S_5 = \{t_5, u_5\}$, the reasoning is analogous, and for for $j = 5$ the result follows by symmetry.

Suppose $j = 2$. Since $|S_1| = |S_5| = 2$, $|S_0| \geq 2$ and $|S_6| \geq 2$. Recalling that $|S_0| \leq 2$ or $|S_6| \leq 2$, we first consider $|S_0| = 2$. Similar to the previous case, s_1 is the only vertex of B_1 dominated by vertices in S_2. Then, S_1 must be either $\{r_1, t_1\}$, $\{r_1, y_1\}$ or $\{t_1, x_1\}$. Note that, in all the cases, vertices x_0 and y_0 are privately dominated by S_0. Suppose $|S_6| \geq 3$. Let $\{a,b,c\}$ be a subset of distinct vertices of S_6. Then, $S' = (S_{\overline{H}} \setminus \{a,b,c\}) \cup \{s_6, x_6, y_6\}$ is a dominating set of G_{l-5}. We can assume $|S_6| \leq 2$, which implies $|S_6| = 2$. Then, $|S_7| \geq 3$. Observe that, if $S_i \in S_i^2$, then either x_i or y_i is not dominated by S_i. Since S_1 do not dominate x_0 and y_0, we conclude that $S_0 \in S_0^1$ and thus, $|S_{l-1}| \geq 3$. Let $\{a,b,c\}$ and $\{d,e,f\}$ be subsets of distinct vertices of S_7 and S_{l-1}, respectively. Then, $S' = (S_{\overline{H}} \setminus (S_0 \cup S_6 \cup \{a,b,c,d,e,f\})) \cup \{s_{l-1}, u_{l-1}, v_{l-1}, r_0, t_0, r_6, t_6, s_7, x_7, y_7\}$ is a dominating set of G_{l-5}. Now, we consider $|S_0| \geq 3$ and thus $|S_6| = 2$. Again, this implies $|S_7| \geq 3$. Let $\{a,b,c\}$ and $\{d,e,f\}$ be subsets of distinct vertices of S_7 and S_0, respectively. Then, $S' = (S_{\overline{H}} \setminus (S_6 \cup \{a,b,c,d,e,f\})) \cup \{s_0, u_0, v_0, r_6, t_6, s_7, x_7, y_7\}$ is a dominating set of G_{l-5} and the result follows. By symmetry, the result also follows for $j = 4$.

At last, consider $j = 3$. Then, $|S_1| = |S_2| = |S_4| = |S_5| = 2$. Hence, $|S_0| \geq 2$ and $|S_6| \geq 2$. Again, as $|S_0| \leq 2$ or $|S_6| \leq 2$, by symmetry, we have just to consider $|S_0| = 2$. This implies $|S_{l-1}| \geq 2$. Suppose $|S_{l-1}| = 2$. Then, $S_{l-1} = \{r_{l-1}, y_{l-1}\}$ or $S_{l-1} = \{t_{l-1}, x_{l-1}\}$ and, thus, $|S_{l-2}| \geq 3$ and either $\{s_{l-2}, x_{l-2}\} \subseteq S_{l-2}$ or $\{s_{l-2}, y_{l-2}\} \subseteq S_{l-2}$. Moreover, $S_2 = \{r_2, v_2\}$ or $S_2 = \{t_2, u_2\}$ and $\{s_3, u_3\} \subseteq S_3$ or $\{s_3, v_3\} \subseteq S_3$. Therefore, S_4 must be either $\{r_4, t_4\}$, $\{r_4, v_4\}$ or $\{t_4, u_4\}$. All the cases imply $S_5 \in S_5^1$ and then $|S_6| \geq 3$. Let $\{a,b,c\}$ and $\{d,e,f\}$ be subsets of distinct vertices of S_{l-2} and S_6, respectively. Then, $S' = (S_{\overline{H}} \setminus (S_0 \cup S_{l-1} \cup \{a,b,c,d,e,f\})) \cup \{s_{l-2}, u_{l-2}, v_{l-2}, r_{l-1}, t_{l-1}, r_0, t_0, s_6, x_6, y_6\}$ is a dominating set of G_{l-5}.

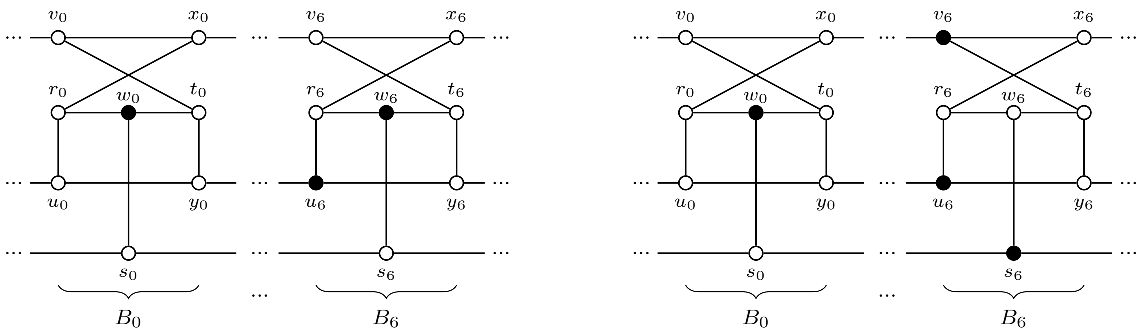
It remains to consider $|S_{l-1}| \geq 3$. Suppose $|S_6| \geq 3$. Let $\{a,b,c\}$ and $\{d,e,f\}$ be subsets of distinct vertices of S_{l-1} and S_6, respectively. Then, $S' = (S_{\overline{H}} \setminus (S_0 \cup \{a,b,c,d,e,f\})) \cup \{s_{l-1}, u_{l-1}, v_{l-1}, r_0, t_0, s_6, x_6, y_6\}$ is a dominating set of G_{l-5}. Finally, suppose $|S_6| = 2$. As we have already seen, if $|S_7| = 2$, then $|S_0| \geq 3$, which contradicts the hypothesis that $|S_0| = 2$. Therefore, $|S_7| \geq 3$. Let $\{a,b,c\}$ and $\{d,e,f\}$ be subsets of distinct vertices of S_{l-1} and S_7, respectively. Then, we conclude that $S' = (S_{\overline{H}} \setminus (S_0 \cup S_6 \cup \{a,b,c,d,e,f\})) \cup \{s_{l-1}, u_{l-1}, v_{l-1}, r_0, t_0, r_6, t_6, s_7, x_7, y_7\}$ is a dominating set of G_{l-5}, completing the proof.

Case 3. $|S_H| = 12$.

If there exist $a, b \in S_0$ such that $a \neq b$ and $a, b \notin \{u_0, v_0\}$, then $S' = (S_{\overline{H}} \setminus \{a,b\}) \cup \{s_0, x_0, y_0\}$ is a dominating set of G_{l-5}. Moreover,

$$|S'| \leq |S_{\overline{H}}| + 1 \leq |S| - |S_H| + 1 \leq |S| - 12 + 1 = |S| - 11. \tag{2}$$

By symmetry, if these distinct vertices a and b belong to S_6 and both are different from x_6 and y_6, the result also follows. Figure 9 exemplifies this case. Note that, by counting, a and b always exist when $|S_0| \geq 4$ or $|S_6| \geq 4$. We can assume $|S_0| \leq 3$ and $|S_6| \leq 3$.



(a) (b)

Figure 9: In (a), $\{u_6, w_6\} \subseteq S_6$; and in (b), these vertices are replaced by $\{s_6, u_6, v_6\}$.

At first, suppose $|S_0| = 3$. If both u_0 and v_0 are not in S_0, then $S' = (S_{\overline{H}} \setminus S_0) \cup \{s_0, x_0, y_0\}$ is a dominating set of G_{l-5} and $|S'| = |S_{\overline{H}}|$. If there is exactly one of $\{u_0, v_0\}$ in S_0, namely a, then $S' = (S_{\overline{H}} \setminus S_0) \cup \{a, s_0, x_0, y_0\}$ is also a dominating set of G_{l-5} and, again, the result follows by (2). Therefore, we can consider $\{u_0, v_0\} \subseteq S_0$. In order to dominate w_0, we can assume, adjusting S_0 if necessary, $s_0 \in S_0$. If there is no vertex of S_6 privately dominated by S_H, then $S' = S_{\overline{H}}$ and the result follows. If there is exactly one of $\{u_6, v_6\}$ privately dominated by S_5, namely a, then $S' = S_{\overline{H}} \cup \{a\}$ dominates G_{l-5} and the result follows once again. Suppose that both u_6 and v_6 are privately dominated by S_5. Hence, $w_6 \in S_6$ since r_6, t_6, x_6 and y_6 do not belong to S_6 and r_6 and t_6 need to be dominated. Thus, $S' = (S_{\overline{H}} \setminus \{w_6\}) \cup \{r_6, t_6\}$ is a dominating set of G_{l-5} and the result follows. We can assume $|S_0| \leq 2$ and, by symmetry, $|S_6| \leq 2$.

Suppose $|S_0| = 2$. Note that S_0 has at least one of $\{u_0, v_0\}$, since otherwise $S' = (S_{\overline{H}} \setminus S_0) \cup \{s_0, x_0, y_0\}$ is a dominating set of G_{l-5} and the result follows by (2). Then, S_0 must be equal to $\{r_0, v_0\}$, $\{t_0, u_0\}$, $\{u_0, w_0\}$ or $\{v_0, w_0\}$. This implies, $|S_{l-1}| \geq 2$. Consider $|S_{l-1}| \geq 3$ and let $\{a, b, c\}$ be distinct vertices of S_{l-1}. Then, $S' = (S_{\overline{H}} \setminus (S_0 \cup \{a, b, c\})) \cup \{s_{l-1}, u_{l-1}, v_{l-1}, s_0, u_0, v_0\}$ is a dominating set of G_{l-5} and the result follows by (2). Suppose $|S_{l-1}| = 2$. Then, $|S_{l-2}| \geq 2$. If $|S_{l-2}| \geq 3$, then the result follows since $S' = (S_{\overline{H}} \setminus (S_0 \cup S_{l-1} \cup S_{l-2})) \cup \{s_{l-2}, u_{l-2}, v_{l-2}, r_{l-1}, t_{l-1}, s_0, x_0, y_0\}$ is a dominating set of G_{l-5}. Then, we may consider $|S_{l-2}| = 2$. This implies $|S_{l-3}| \geq 2$. If $|S_{l-3}| \geq 3$, then the result follows again, since $S' = (S_{\overline{H}} \setminus (S_0 \cup S_{l-1} \cup S_{l-2} \cup S_{l-3})) \cup \{s_{l-3}, u_{l-3}, v_{l-3}, r_{l-2}, t_{l-2}, r_{l-1}, t_{l-1}, s_0, x_0, y_0\}$ is a dominating set of G_{l-5}. Consider $|S_{l-3}| = 2$. By symmetry, we conclude that, if $|S_6| = 2$, then $|S_7| = |S_8| = |S_9| = 2$.

Since $|S_0| = 2$ and $(|S_{l-3}| + |S_{l-2}| + |S_{l-1}|) = 6$, we conclude that $|S_1| \geq 3$. Moreover, $|S_5| \geq 2$ as $|S_6| \geq 2$. Suppose that exists some j, $2 \leq j \leq 4$, such that $|S_j| = 1$. Initially, consider $j = 2$. Note that all possible sets that form S_0 do not dominate one of $\{x_0, y_0\}$. Thus, if $|S_2| = 1$, then $\{s_2, x_2, y_2\} \subseteq S_2$ and $|S_1| \geq 4$. Therefore, $|S_3| = 3$ and $|S_4| = |S_5| = 2$. Then, $|S_6| = 2$, since $|S_6| = 1$ implies $|S_5| \geq 3$. However, as $|S_6| = 2$, we have $|S_7| = |S_8| = |S_9| = 2$, which forms a prohibited configuration with S_5. Therefore, $j \neq 2$. Suppose $j = 3$. Then, $|S_1| = |S_2| = |S_4| = 3$ and $|S_5| = 2$. Again, this implies $|S_6| = 2$ and we have a prohibited configuration. Then, $j \neq 3$. Finally, consider $j = 4$. This implies $|S_1| = |S_3| = |S_5| = 3$ and $|S_2| = 2$. This implies $|S_6| = 2$. Since one of $\{u_6, v_6\}$ must be dominated by S_5 and $|S_4| = 1$, we conclude that $|S_5| \geq 4$, a contradiction. We conclude that $|S_j| \geq 2$ for every $1 \leq j \leq 5$.

Note that $|S_1| \leq 4$. If $|S_1| = 4$, then $|S_j| = 2$ for every $2 \leq j \leq 5$. Hence, $|S_6| = 2$ and we have a prohibited configuration. Consider $|S_1| = 3$. Furthermore, $|S_5| = 3$, since $|S_6| = 1$ implies $|S_5| \geq 3$ so as to not have a prohibited configuration when $|S_6| = 2$. Since $|S_{l-3}| = |S_{l-2}| = |S_{l-1}| = |S_0|$, we have either $S_0 = \{r_0, v_0\}$ or $S_0 = \{t_0, u_0\}$. Suppose $S_0 = \{r_0, y_0\}$. As s_0 and y_0 are not dominated by $S_0 \cup S_{l-1}$, we have $\{s_1, u_1\} \subseteq S_1$. Also, as $|S_1| = 3$, we must have one of $\{v_1, t_1\}$ in order to dominate both v_1 and t_1. Therefore, $x_1, y_1 \notin S_1$ and thus u_2 and v_2 must be dominated by S_2. We conclude that $S_2 \in S_2^1$. In particular, S_2 is equal to $\{r_2, t_2\}$, $\{r_2, v_2\}$ or $\{t_2, u_2\}$. Then, u_3 and v_3 must be dominated by S_3 and $S_3 \in S_3^1$. However, if $S_3 \in S_3^1$, then either $|S_2| \geq 3$ or $|S_4| \geq 3$, a contradiction. The case when $S_0 = \{t_0, u_0\}$ is analogous.

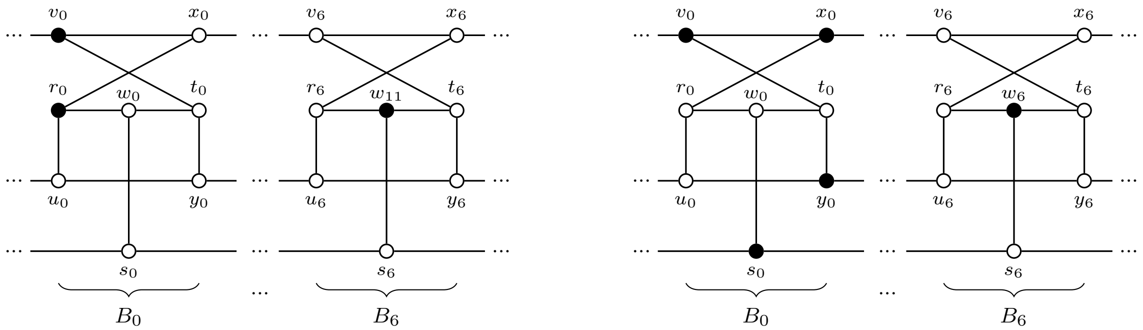
It remains to consider $|S_0| = 1$. By symmetry, $|S_6| = 1$. Then, $|S_1| = |S_5| = 3$ and $|S_2| = |S_3| = |S_4| = 2$. Also, $S_1 = \{s_1, u_1, v_1\}$. Similar to previous arguments, we conclude that $S_2 \in S_2^1$ and $S_3 \in S_3^1$. However, this is a contradiction, since $S_3 \in S_3^1$ implies either $|S_2| \geq 3$ or $|S_4| \geq 3$.

Case 4. $|S_H| = 13$.

Suppose $|S_6| = 1$. Then, $S_6 = \{w_6\}$. Notice that, if s_0 is not privately dominated by $s_1 \in S_1$, then $S_{\overline{H}}$ dominates s_0 and $S' = S_{\overline{H}} \cup \{x_0, y_0\}$ dominates G_{l-5}. Then,

$$|S'| \leq |S_{\overline{H}}| + 2 \leq |S| - |S_H| + 2 \leq |S| - 13 + 2 = |S| - 11. \tag{3}$$

We can assume that s_0 is privately dominated by $s_1 \in S_1$. Thus, $s_0 \notin S_0$ and $w_0 \notin S_0$. Moreover, in order to dominate w_0 we must have either $r_0 \in S_0$ or $t_0 \in S_0$. Let $a \in (S_0 \cap \{r_0, t_0\})$. Then, $S' = (S_{\overline{H}} \setminus \{a\}) \cup \{s_0, x_0, y_0\}$ is a dominating set of G_{l-5} satisfying (3). Figure 10 exemplifies this case.



(a) (b)

Figure 10: In (a), $\{r_0, v_0\} \subseteq S_0$; and in (b), r_0 is replaced by $\{s_0, x_0, y_0\}$.

In order to conclude this case, suppose $|S_6| \geq 2$. Note that, if $|S_6| = 2$, then $S_6 \neq \{x_6, y_6\}$. Therefore, there is at least one vertex $a \in S_6$ such that $a \neq x_6$ and $a \neq y_6$. Thus, $S' = (S_{\overline{H}} \setminus \{a\}) \cup \{s_6, u_6, v_6\}$ is a dominating set of G_{l-5} satisfying (3).

Case 5. $|S_H| \geq 14$.

Let $S' = S_{\overline{H}} \cup \{s_6, u_6, v_6\}$. Then, S' is a dominating set of G_{l-5}. Moreover, we have

$$|S'| \leq |S_{\overline{H}}| + 3 \leq |S| - |S_H| + 3 \leq |S| - 14 + 3 = |S| - 11$$

and the result follows. □

Finally, we can show that $\gamma(G_l) \leq \lceil \frac{11l}{5} \rceil$ using the previous lemma and establish our main result in Theorem 3.

Theorem 3. *Let G_l be a Goldberg Graph with $l \geq 3$. Then,* $\gamma(G_l) = i(G_l) = \lceil \frac{11l}{5} \rceil$.

Proof. By Theorem 1, $\gamma(G_l) \leq \lceil \frac{11l}{5} \rceil$ and, by Theorem 2, $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$ for $l \in \{3,4,5,6,7\}$. Now, we prove that $\gamma(G_l) \geq \lceil \frac{11l}{5} \rceil$ for $l \geq 8$. Let $\mathcal{F} = \{G_l : \gamma(G_l) < \lceil \frac{11l}{5} \rceil\}$. Note that G_l, $l \in \{3,4,5,6,7\}$, does not belong to $\mathcal{F}$, that is, if $G_l \in \mathcal{F}$, then $l \geq 8$. Suppose $\mathcal{F} \neq \emptyset$. Let $G_l \in \mathcal{F}$ be the graph with minimum $|V(G_l)|$. Since $l \geq 8$, by Lemma 2 we have

$$\gamma(G_{l-5}) \leq \gamma(G_l) - 11 < \lceil \tfrac{11l}{5} \rceil - 11 < \lceil \tfrac{11(l-5)}{5} \rceil.$$

Hence, $G_{l-5} \notin \mathcal{F}$ when $(l-5) \leq 7$ and $G_{l-5} \in \mathcal{F}$ otherwise. However, $|V(G_{l-5})| < |V(G_l)|$, which contradicts the choice of G_l. Therefore, $\mathcal{F} = \emptyset$. Then, $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$.

In order to conclude the proof, we must show that $i(G_l) = \lceil \frac{11l}{5} \rceil$. By Theorem 1, $i(G_l) \leq \lceil \frac{11l}{5} \rceil$. Furthermore, from $\gamma(G_l) \leq i(G_l)$ and $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$, we obtain $i(G_l) = \lceil \frac{11l}{5} \rceil$. □

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