Matching extendability in cartesian product of hypercubes and paths

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Technical Report - IC-25-02 - Relatório Técnico
October - 2025 - Outubro

UNIVERSIDADE ESTADUAL DE CAMPINAS
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Matching extendability in cartesian product of hypercubes and paths[*]

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Abstract

A matching M in a graph G is said to be extendable to a perfect matching if there exists a perfect matching M^* of G such that $M \subseteq M^*$. In this work, we study the extendability of matchings under a neighbourhood condition: no unsaturated vertex has all of its neighbours M-saturated. Vandenbussche and West showed that, in the hypercube Q_n, any matching of size at most $2n-4$ is extendable to a perfect matching if and only if it satisfies this condition. We extend their result to the cartesian product $Q_n \,\square\, P_m$ by proving that every matching of size at most $2n-2$ is extendable to a perfect matching if and only if it does not saturate the neighbourhood of any unsaturated vertex. Furthermore, we demonstrate that this bound is tight by constructing a matching of size $2n+2\delta(H)-3$ in $Q_n \,\square\, H$ that satisfies the neighbourhood condition but is not extendable to a perfect matching.

1 Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $N_G(v) = \{u \in V(G) : uv \in E(G) \text{ and } u \neq v\}$ be the *(open) neighbourhood* of a vertex $v \in V(G)$. Each $u \in N_G(v)$ is called a *neighbour* of v. Similarly, for a subset $S \subseteq V(G)$, the *neighbourhood* of S is given by $N_G(S) = \cup_{v \in S} N_G(v)$. When the context is clear, we omit the subscript and simply write $N(v)$ and $N(S)$. The *degree* of a vertex $v \in V(G)$ is $d(v) = |N(v)|$. The *minimum degree* of G is denoted by $\delta(G) = \min_{v \in V(G)}\{d(v)\}$ and the *maximum degree* by $\Delta(G) = \max_{v \in V(G)}\{d(v)\}$. For $u, v \in V(G)$, the *distance* $\operatorname{dist}(u,v)$ between u and v is the number of edges in the shortest path connecting them; $\operatorname{dist}(u,v) = \infty$ if such a path does not exist. A *matching* M in G is a set of pairwise non-adjacent edges. A vertex $v \in V(G)$ is said to be *M-saturated* if it is incident with some edge in M; otherwise, v is said to be *M-unsaturated*; we drop M when it is clear in the context. The matching M is called *perfect* when every vertex of G is M-saturated and we say that M *extends* to a perfect matching if M is a subset of some perfect matching of G.

Matching theory has played a fundamental role in Graph Theory since its origins in the late 19th century. Over the years, significant progress has been made in characterizing their existence and enumerating perfect matchings [27, 29], and also analysing their interaction with other structural properties of graphs [20, 28]. In particular, numerous studies have introduced decomposition techniques for graphs in terms of their matchings [7, 8, 12].

In one of such study, Lovász [18] introduced the concept of a bicritical graph. A graph is *bicritical* if $G-u-v$ has a perfect matching for every $u, v \in V(G)$, $u \neq v$. Motivated by this

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[*] This research was supported by CAPES, grants 88887.520851/2020-00 and 88887.600764/2021-00, and by DAAD, grant 91882084.

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concept, Plummer [22] introduced in 1980 the notion of k-extendable graphs. Let G be a graph on $2p$ vertices and let k be an integer such that $1 \leq k \leq p-1$. Then, G is said to be *k-extendable* if every matching of size k in G extends to a perfect matching of G. Note that, for $k = 1$, this definition is equivalent to stating that G is a *matching-covered graph* (a graph such that every edge belongs to some perfect matching), which has been extensively studied in the literature [6, 19]. In the same article, Plummer [22] showed that every 2-extendable graph is either bipartite or a 3-connected bicritical graph. Furthermore, the author proved that every k-extendable graph is also $(k-1)$-extendable and $(k+1)$-connected.

Since then, the theory of k-extendability has been further developed through numerous contributions, addressing necessary and sufficient conditions for k-extendable graphs [15], their behaviour on surfaces [2] and its criticality [3]. Additionally, several studies have addressed the relationship between extendability and graph parameters such as independence number, binding number, toughness, among others [5, 17, 21, 32]. A comprehensive overview of the foundational concepts and early developments in the theory of k-extendable graphs can be found in three extensive surveys authored by Plummer [24, 25, 26].

In 1998, Lakhal and Litzler [13] showed that deciding whether a bipartite graph is k-extendable can be done in polynomial time. For an arbitrary graph, Hackfeld and Koster [10] proved that the problem belongs to co-NP-complete. This naturally leads to the study of k-extendability in classes of graphs. Some examples include regular graphs [1], triangle-free graphs [16] and planar graphs [23]. In addition, the problem has been investigated for graphs obtained from certain operations, such as cartesian product. For instance, Györi and Plummer [9] proved that the cartesian product of a k-extendable and an l-extendable graph is a $(k+l+1)$-extendable graph. Other operations, such as the lexicographic product [4] and the complementary prism of graphs [11], have also been explored.

In the study of k-extendability of graphs, several properties have been shown to be useful. Among them, the neighbourhood of unsaturated vertices is often analysed, as it provides immediate insight into the matching extendability of graphs.

Property 1. *Let G be a graph and M a non-perfect matching of G. If there exists an unsaturated vertex $u \in V(G)$ such that every neighbour $v \in N(u)$ is M-saturated, then M is not extendable to a perfect matching of G.*

Some studies have investigated the extendability of matchings in the absence of this type of neighbourhood obstruction. Limaye and Sarvate [14] proved that in a hypercube Q_n, $n \geq 2$, a matching M of size at most n can be extended to a perfect matching of Q_n if and only if M does not saturate the neighbourhood of any unsaturated vertex. Since any matching M of Q_n with $1 \leq |M| \leq n-1$ does not saturate the neighbourhood of any unsaturated vertex, this result implies that the hypercube is k-extendable for all $1 \leq k \leq n-1$. The authors also showed that, for the hypercube Q_4, there exists a matching of size $n+1 = 5$ that does not saturate the neighbourhood of any unsaturated vertex and does not extend to a perfect matching. For $n \geq 5$, Vandenbussche and West [30] strengthened these results for matchings of size up to $2n-4$, as stated in Theorem 2. More recently, the problem was revisited by Vandenbussche and Westlund [31], who extended the analysis to the n-fold cartesian product of cycles $G(2m)_d \cong C_{2m} \,\square\, ... \,\square\, C_{2m}$.

Theorem 2 (Vandenbussche and West [30]). *Let M be a matching of the hypercube Q_n, $n \geq 5$, such that $|M| \leq 2n-4$. Then, M is extendable to a perfect matching of Q_n if and only if M does not saturate the neighbourhood of any unsaturated vertex.* $\square$

In our work, we focus on the cartesian product $Q_n \,\square\, P_m$, $n \geq 4$ and $m \geq 2$, and prove that a matching of size at most $2n-2$ is extendable to a perfect matching if and only if it does not saturate the neighbourhood of any unsaturated vertex. Furthermore, we show that this bound is tight. We also provide some insights about the extendability of matchings that satisfy the neighbourhood condition for the cartesian product of hypercubes with arbitrary graphs.

2 Results

An n-dimensional hypercube, with $n \geq 1$, denoted by Q_n, is the graph whose vertex set consists of the 2^n binary strings of length n. That is, if $v \in V(Q_n)$, then $v = b_1 b_2 \dots b_n$ such that every *coordinate* $b_i \in \{0,1\}$, $1 \leq i \leq n$. Moreover, two vertices are adjacent if and only if their strings differ in exactly one coordinate. For a positive integer i, $1 \leq i \leq n$, we define the *i-th decomposition* of Q_n such that $V(Q_n) = \{V_0^i, V_1^i\}$, with V_0^i containing all vertices with 0 in coordinate i, and V_1^i containing all vertices with 1 in coordinate i. Note that $Q_n[V_0^i]$ and $Q_n[V_1^i]$ are both isomorphic to the hypercube Q_{n-1}.

Given the cartesian product $G \cong Q_n \,\square\, H$, we define the *canonical notation* of G as follows. Let $V(Q_n) = \{v_0, v_1, \dots, v_{2^n-1}\}$, such that the index of each vertex v_j, $0 \leq j \leq 2^n - 1$, corresponds to the decimal value of the binary representation of its binary string. Let $V(H) = \{u_1, u_2, \dots, u_m\}$. We construct the graph G from m copies of the hypercube Q_n, denoted $G^1, G^2, \dots, G^m$. Moreover, we define the vertex set of each copy G^i as $V(G^i) = \{v_1^i, v_2^i, \dots, v_{2^n}^i\}$, for $1 \leq i \leq m$. These copies are connected in such a way that the induced subgraph $G[v_j^1, v_j^2, \dots, v_j^m]$ is isomorphic to the graph H, for every $1 \leq j \leq 2^n$. We say that v_j^i and v_j^k, $i \neq k$, are *corresponding vertices*.

Given the copies $G^1, G^2, \dots, G^m$ of Q_n in the canonical notation, we can apply the i-th decomposition of each G^k, such that $V(G^k) = \{V_0^{k,i}, V_1^{k,i}\}$. Thus, $G[\cup_{k=1}^m V_1^{k,i}] \cong G[\cup_{k=1}^m V_2^{k,i}] \cong Q_{n-1} \,\square\, H$. Observe that, by the definition of the hypercube, $G \cong (Q_{n-1} \,\square\, H) \,\square\, K_2$. In other words, G can be built from two subgraphs G' and G'', both isomorphic to $Q_{n-1} \,\square\, H$, such that G' and G'' are connected by a perfect matching. Finally, since there are n possible i-th decompositions of each G^k, we conclude that for any two distinct vertices u and v in G, if they are not corresponding vertices, then it is possible to built G in such a way that u belongs to G' and v belongs to G''.

Vandenbussche and West [30] showed that the bound in Theorem 2 is tight; that is, for every $n \geq 4$, there exists a matching in Q_n of size $2n-3$ that does not saturate the neighbourhood of any unsaturated vertex and cannot be extended to a perfect matching. The idea behind the construction involves the i-th decomposition of Q_n. Let $Q^k \cong Q_{n-1}$, $k \in \{1,2\}$, be the two $(n-1)$-dimensional hypercubes obtained in this decomposition. Let u and v be two vertices in Q^1 such that $d(u,v) = 2$. Note that $|N(u) \cap N(v) \setminus \{u,v\}| = 2$. Let $N(u) \cap N(v) = \{x,y\}$. Let $R = N(u) \setminus \{x,y\}$ and $S = N(v) \setminus \{x,y\}$. The authors argue that it is possible to construct a matching M with $2n-3$ edges that saturates all vertices in $R \cup S \cup \{x\}$.

In our first result, presented in Corollary 3, we adapted the construction proposed by Vandenbussche and West [30] to build non-extendable matchings in other graph classes related to hypercubes. Consider graph $G \cong Q_n \,\square\, H$, with $n \geq 4$ and $|V(H)| = m$, using the canonical notation. By applying the construction of Vandenbussche and West, it is possible to obtain a matching of size $2n-3$ in a copy G^i that does not saturate the neighbourhood of any unsaturated vertex. Let $w \in V(H)$ such that $d(w) = \delta(H)$ and let G^i be the copy of Q_n corresponding to w. Let u, v and y be the vertices as in the aforementioned construction. Note that $\{u,v,y\} \subseteq V(G^i)$. Since u and v have each $\delta(H)$ neighbours in $V(G) \setminus V(G^i)$, we can obtain a matching in $G[V(G) \setminus V(G^i)]$ with size $2\delta(H)$ that does not saturate the neighbourhood of any unsaturated vertex. Again, u and v can only be matched with y and thus there is no perfect matching in G that contains M.

Corollary 3. *Let $G \cong Q_n \,\square\, H$ with $n \geq 4$. Then, there exists a matching M in G with*

$|M| = 2n + 2\delta(H) - 3$ such that M does not saturate the neighbourhood of any unsaturated vertex and M is not extendable to a perfect matching of G. $\square$

In the special case in which H is a path P_m, the bound provided by Corollary 3 implies that G admits a matching of size $2n - 1$ that does not saturate the neighbourhood of any unsaturated vertex and is not extendable to a perfect matching. We shall prove that this bound is tight for the graph $Q_n \,\square\, P_m$, $n \geq 4$ and $m \geq 2$, by showing that every matching of size at most $2n-2$ that does not saturate the neighbourhood of any unsaturated vertex is extendable to a perfect matching.

Our main result, presented in Theorem 7, states that a matching M in a graph $G \cong Q_n \,\square\, P_m$, $n \geq 4$ and $m \geq 2$, with $|M| \leq 2n - 2$, is extendable to a perfect matching if and only if M does not saturate the neighbourhood of any unsaturated vertex. In order to prove it, we have to establish auxiliary lemmas that use the well-known Hall's Theorem: in a bipartite graph with bipartition $\{X, Y\}$, there exists a perfect matching saturating X if and only if $|N(S)| \geq |S|$ for every subset $S \subseteq X$. We show that Hall's condition remains satisfied even after the removal of the vertices saturated by the initial matching. In particular, we show that the surplus $|N(S)| - |S|$ is sufficiently large to guarantee the existence of a perfect matching in the remaining graph.

We proceed by showing the first auxiliary lemma, where we establish a bound on the neighbourhood sizes when the set S is small and contained in one part of the bipartition.

Lemma 4. *Let $G \cong Q_n \,\square\, P_m$, $n \geq 3$ and $m \geq 2$, and let $\{X, Y\}$ be a bipartition of G. Let $S \subseteq X$. If $2 \leq |S| \leq n$, then $|N(S)| - |S| \geq 2n - 2$.* $\square$

Lemma 6 establishes bounds for the neighbourhood of larger vertex sets, using the following proposition.

Proposition 5. *Let G be a graph and let $\{X, Y\}$ be a bipartition of G such that $|X| = |Y|$. Then, for every $S \subseteq X$, $|S| - |N(S)| \leq |T| - |N(T)|$, with $T = Y \setminus N(S)$.* $\square$

Lemma 6. *Let $G \cong Q_n \,\square\, P_m$, $m, n \geq 3$, and let $\{X, Y\}$ be a bipartition of G. For $S \subseteq X$, if $2 \leq |S| \leq m2^{n-1} - 2n + 1$, then $|N(S)| - |S| \geq 2n - 2$.*

Proof. Let G, $\{X, Y\}$ and S be as stated in the hypothesis. Suppose $2 \leq |S| \leq m2^{n-1} - 2n + 1$. By Lemma 4, if $2 \leq |S| \leq n$, the claim holds. Suppose $|S| > n$. First, suppose $|S| \leq m2^{n-2}$. Note that, by the definition of S, it holds that $N(S) \cap S = \emptyset$. Also, since G is a bipartite graph that admits a perfect matching, by Hall's Theorem, we have that $|N(S)| \geq |S|$ for every $S \subseteq X$. Therefore, $|N(S)| - |S| \geq 0$.

We proceed by induction on n. Consider $n = 3$. We show that $|N(S)| - |S| \geq 4$. Assume G with its canonical notation and let $S_{G^i} = S \cap G^i$. Since $S \subseteq X$, it follows that $|S_{G^i}| \leq 4$, for every $1 \leq i \leq m$. Furthermore, if $|S_{G^i}| = 1$, then $|N_{G^i}(S_{G^i})| - |S_{G^i}| = 2$; if $|S_{G^i}| = 2$, then $|N_{G^i}(S_{G^i})| - |S_{G^i}| = 2$; if $|S_{G^i}| = 3$, then $|N_{G^i}(S_{G^i})| - |S_{G^i}| = 1$; and if $|S_{G^i}| = 4$, then $|N_{G^i}(S_{G^i})| - |S_{G^i}| = 0$. Observe that $|N(S)| - |S| \geq \sum_{i=1}^{m} |N_{G^i}(S_{G^i})| - |S_{G^i}|$, since the sum of the neighbourhoods within the isolated copies does not account for inter-copy neighbours.

Suppose $|S_{G^i}| \neq \emptyset$ for every $1 \leq i \leq m$. Since $m \geq 3$ and $|S| \leq 2m$, we conclude that there exist at least two distinct sets S_{G^i} and S_{G^j} such that $1 \leq |S_{G^i}| \leq 2$ and $1 \leq |S_{G^j}| \leq 2$. Hence, $|N_{G^i}(S_{G^i})| - |S_{G^i}| + |N_{G^j}(S_{G^j})| - |S_{G^j}| \geq 4$. As $|N(S)| - |S| \geq \sum_{i=1}^{m} |N_{G^i}(S_{G^i})| - |S_{G^i}| \geq 4$, the result follows. This conclusion is used implicitly in the remainder of the proof.

Now, suppose that there exists i such that $S_{G^i} = \emptyset$. Choose G^i such that $S_{G^i} = \emptyset$ and i is maximum. We can assume, without loss of generality, that G^{i+1} exists (adjust notation if necessary) and that $S_{G^{i+1}} \neq \emptyset$. There are $|S_{G^{i+1}}|$ neighbours of $S_{G^{i+1}}$ in G^i, that is, $|N_{G^i}(S_{G^{i+1}})| = |S_{G^{i+1}}|$. So, if $|S_{G^{i+1}}| \geq 2$, we conclude that $|N(S)| - |S| \geq |N_{G^{i+1}}(S_{G^{i+1}})| - |S_{G^{i+1}}| + |N_{G^i}(S_{G^{i+1}})| \geq 4$.

Consider $|S_{G^{i+1}}| = 1$. Then, $|N_{G^{i+1}}(S_{G^{i+1}})| - |S_{G^{i+1}}| + |N_{G^i}(S_{G^{i+1}}) \geq 3$. This implies that, if there exists j, for $1 \leq j \leq m$ and $j \notin \{i, i+1\}$, such that $1 \leq |S_{G^j}| \leq 3$, then $|N(S)| - |S| \geq (|N_{G^{i+1}}(S_{G^{i+1}})| - |S_{G^{i+1}}| + |N_{G^i}(S_{G^{i+1}})) + (|N_{G^j}(S_{G^j})| - |S_{G^j}|) \geq 4$ and the result follows. Consider then $|S_{G^j}| \in \{0, 4\}$ for every $j \notin \{i, i+1\}$. Suppose $i + 1 = m$. Since $m \geq 3$ and $|S| \geq 4$, there exists j, for $1 \leq j \leq i - 1$, such that $|S_{G^j}| = 4$. Choose G^j such that $|S_{G^j}| = 4$ and j is maximum. Thus, $S_{G^{j+1}} = \emptyset$. Therefore, $|N_{G^j}(S_{G^j})| - |S_{G^j}| + |N_{G^{j+1}}(S_{G^j})| \geq 4$ and the result follows. Suppose $i + 1 < m$. If $S_{G^{i+2}} = \emptyset$, then $|N_{G^i}(S_{G^{i+1}})| + |N_{G^{i+1}}(S_{G^{i+1}})| - |S_{G^{i+1}}| + |N_{G^{i+2}}(S_{G^{i+1}})| \geq 4$ and the result follows. Suppose $|S_{G^{i+2}}| = 4$. Since each vertex of $S_{G^{i+2}}$ has a distinct neighbour in G^{i+1} and $|S_{G^{i+1}}| = 1$, there must be a vertex of $N(S_{G^{i+2}})$ that is not adjacent to the vertex of $S_{G^{i+1}}$. Therefore, $|N_{G^{i+1}}(S_{G^{i+1}})| - |S_{G^{i+1}}| + |N_{G^{i+2}}(S_{G^{i+2}})| - |S_{G^{i+2}}| + |N_{G^{i+1}}(S_{G^{i+2}})| \geq 4$. This concludes the base case.

Consider $n \geq 4$. Suppose that G cannot be partitioned into $H_1 \cong Q_{n-1} \square P_m$ and $H_2 \cong Q_{n-1} \square P_m$ such that $S \cap V(H_1) \neq \emptyset$ and $S \cap V(H_2) \neq \emptyset$. Hence, each G^i has at most one vertex in S and, for every $\{u, v\} \subseteq S$, u and v are corresponding vertices. Since every $v^i_k \in S$ has n neighbours in G^i and $|S| \geq 5$, we conclude that $|N(S)| - |S| \geq 2n - 2$.

Suppose that there exists a partition of G into H_1 and H_2 such that $S \cap V(H_1) \neq \emptyset$ and $S \cap V(H_2) \neq \emptyset$. Let $S_i = S \cap V(H_i)$, $i \in \{1, 2\}$. We can assume, without loss of generality, $|S_1| \geq |S_2|$. As $|S| \geq 5$, we have $|S_1| \geq 3$. Suppose $|S_1| \leq m2^{n-2} - 2(n-1) + 1$. By the induction hypothesis, $|N_{H^1}(S_1)| - |S_1| \geq 2(n-1) - 2$. To conclude the result, we analyse the cases $|S_2| = 1$ and $|S_2| \geq 2$.

First, suppose $|S_2| = 1$. Then, $|N_{H^2}(S_2)| \geq (n-1)+1$. Since $|N(S)| - |S| \geq (|N_{H^1}(S_1)| - |S_1|) + (|N_{H^2}(S_2)| - |S_2|)$, we conclude that $|N(S)| - |S| \geq 3n-4$. As $n \geq 4$, it holds that $3n-4 \geq 2n-2$ and the result follows. Suppose $|S_2| \geq 2$. By the induction hypothesis, $|N_{H^2}(S_2)| - |S_2| \geq 2(n-1) - 2$. Similarly, we have $|N(S)| - |S| \geq 4n - 8 \geq 2n - 2$.

Now, suppose $|S_1| \geq m2^{n-2} - 2(n-1) + 2 = m2^{n-2} - 2n + 4$. Let $S'_1 \subseteq S_1$ such that $|S'_1| = m2^{n-2} - 2(n-1) + 1 = m2^{n-2} - 2n + 3$. By the induction hypothesis, $|N_{H^1}(S'_1)| - |S'_1| \geq 2(n-1) - 2 = 2n - 4$. Then,

$$\begin{aligned}
|N_{H^1}(S'_1)| - |S'_1| &\geq 2n - 4 \\
|N_{H^1}(S'_1)| &\geq |S'_1| + 2n - 4 \\
&\geq m2^{n-2} - 2n + 3 + 2n - 4 \\
&= m2^{n-2} - 1.
\end{aligned}$$

Since $|N_{H^1}(S_1)| \geq |N_{H^1}(S'_1)|$, it follows that $|N_{H^1}(S_1)| \geq m2^{n-2} - 1$. Moreover, note that each vertex of S_1 has a neighbour in H^2. Therefore, $|N(S)| \geq |N_{H^1}(S_1)| + |S_1| \geq m2^{n-2} - 1 + |S_1|$. Since $|S_1| = |S| - |S_2|$, we have $|N(S)| \geq m2^{n-2} - 1 + |S| - |S_2|$, which implies $|N(S)| - |S| \geq m2^{n-2} - 1 - |S_2|$.

If $|S_2| \leq m2^{n-2} - 2n + 1$, then $|N(S)| - |S| \geq 2n - 2$ as required. Suppose $|S_2| \geq m2^{n-2} - 2n + 2$. Since $|S| \leq m2^{n-2}$, $|S_1| \geq m2^{n-2} - 2n + 4$ and $|S_2| = |S| - |S_1|$, it follows that $|S_2| \leq 2n - 4$. Thus, we obtain $2n - 4 \geq |S_2| \geq m2^{n-2} - 2n + 2$, which implies $4n - 6 \geq m2^{n-2}$. First, observe that $4n - 6 < m2^{n-2}$ for $n \geq 5$ and $m \geq 3$. Since by hypothesis $n \geq 4$, we must have $n = 4$, which implies that the inequality $4n - 6 \geq m2^{n-2}$ is satisfied only when $m \leq 2$. However, this contradicts the fact that $m \geq 3$.

Now, consider $|S| > m2^{n-2}$. Let $T = Y \setminus N(S)$. Then, $|T| = |Y| - |Y \cap N(S)|$ and, since $N(S) \subseteq Y$, it follows that $|T| = |Y| - |N(S)|$. As $|Y| = m2^{n-1}$ and $|N(S)| \geq |S| > m2^{n-2}$, we conclude that $|T| < m2^{n-2}$. By Proposition 5, $|S| - |N(S)| \leq |T| - |N(T)|$ and thus $|N(S)| - |S| \geq |N(T)| - |T|$. To conclude the proof, we consider the cases where $|T| \leq 1$ and $|T| \geq 2$.

Suppose $|T| \leq 1$. As $|T| = |Y| - |N(S)|$, we conclude that $|N(S)| \geq m2^{n-1} - 1$. Since, by hypothesis, $|S| \leq m2^{n-1} - 2n + 1$, it follows that $|N(S)| - |S| \geq 2n - 2$. Suppose $|T| \geq 2$. Since $|T| < m2^{n-2}$ and $m, n \geq 3$, we have $2 \leq |T| \leq m2^{n-1} - 2n + 1$. Therefore, as we proved before, $|N(T)| - |T| \geq 2n - 2$. Hence $|N(S)| - |S| \geq |N(T)| - |T| \geq 2n - 2$. This concludes the proof. $\square$

Using the previous lemmas, we prove now our main result.

Theorem 7. *Let $G \cong Q_n \square P_m$, $n \geq 4$ and $m \geq 2$. Let M be a matching of G such that $|M| \leq 2n - 2$. Then, M is extendable to a perfect matching of G if and only if M does not saturate the neighbourhood of any unsaturated vertex.*

Proof. Suppose that M is extendable to a perfect matching of G. Then, by Property 1, M does not saturate the neighbourhood of any unsaturated vertex. Conversely, suppose M, with $|M| \leq 2n - 2$, does not saturate the neighbourhood of any unsaturated vertex. For $m = 2$, as $G \cong Q_n \square P_m \cong Q_{n+1}$, we conclude, by Theorem 2, that M is extendable to a perfect matching of G. Consider $m \geq 3$.

Let V_M be the set of M-saturated vertices of G and let $\{X, Y\}$ be a bipartition of G. To prove the result, we show that $G[V(G) \setminus V_M]$ admits a perfect matching. It suffices, by Hall's Theorem, to show that $|N(S) \setminus V_M| \geq |S \setminus V_M|$ for every $S \subseteq X$.

Since M does not saturate the neighbourhood of any unsaturated vertex, if follows that $|N(S) \setminus V_M| \geq 1$. Therefore, if $|S| = 1$, the result follows. Suppose $2 \leq |S| \leq m2^{n-1} - 2n + 1$. Then, by Lemma 6, $|N(S)| - |S| \geq 2n - 2$ and thus $|N(S)| \geq |S| + 2n - 2$. On the other hand, $N(S) \subseteq Y$ and $|N(S)| = |N(S) \cap V_M| + |N(S) \setminus V_M|$. Note that $|V_M \cap Y| = |M| \leq 2n - 2$ and $|N(S) \cap V_M| \leq |V_M \cap Y|$. Thus, we have $|N(S) \setminus V_M| \geq |S|$. As $|S| \geq |S \setminus V_M|$, the result follows.

Suppose $|S| \geq m2^{n-1} - 2n + 2$. Let $S' \subseteq S$ such that $|S'| = m2^{n-1} - 2n + 1$. Then, by Lemma 6, it follows that $|N(S')| \geq |S'| + 2n - 2 = m2^{n-1} - 1$. Since $S' \subseteq S$, $|N(S)| \geq m2^{n-1} - 1$. Let $T = Y - N(S) - (V_M \cap Y)$. As $T \subseteq Y \setminus N(S)$ and $|Y| = m2^{n-1}$, we conclude that $|T| \leq m2^{n-1} - (m2^{n-1} - 1) = 1$. Moreover, since M does not saturate the neighbourhood of any unsaturated vertex, $|N(T) \setminus V_M| \geq 1 \geq |T|$. Therefore, $|N(T) \setminus V_M| - |T| \geq 0$.

Let $G' = G \setminus V_M$. Note that $N_{G'}(T) = N(T) \setminus V_M$. Then, $|N_{G'}(T)| - |T| = |N(T) \setminus V_M| - |T| \geq 0$. Since G' is bipartite , by Proposition 5, we have $|N_{G'}(S)| - |S \setminus V_M| \geq |N_{G'}(T)| - |T| \geq 0$. Since $N_{G'}(S) = N(S) \setminus V_M$, it follows that $|N(S) \setminus V_M| - |S \setminus V_M| \geq 0$ and, therefore, $|N(S) \setminus V_M| \geq |S \setminus V_M|$. $\square$

3 Concluding remarks

The results presented in this work shed light on the extendability of matchings that do not saturate the neighbourhood of unsaturated vertices in cartesian products involving hypercubes. In particular, for $Q_n \square H$ when H is an arbitrary graph, we proved in Corollary 3 that there exists a matching of size $2n + 2\delta(H) - 3$ that satisfies the neighbourhood condition but is not extendable to a perfect matching. This motivates the following conjecture.

Conjecture 8. *Let $G \cong Q_n \square H$, $n \geq 4$, and let M be a matching of G such that $|M| \leq 2n + 2\delta(H) - 4$. Then, M is extendable to a perfect matching of G if and only if M does not saturate the neighbourhood of any unsaturated vertex.*

We have verified this conjecture when H is isomorphic to a path. The techniques used in this case are based on Hall's Theorem and structural properties of the hypercube, suggesting that the results may be generalized to cases where H is a bipartite graph. However, further efforts are needed to establish the conjecture for an arbitrary graph H.

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