

# Extending unlocked matchings in graphs

*A. A. Pereira*      *C. N. Campos*

Technical Report - IC-25-03 - Relatório Técnico  
October - 2025 - Outubro

UNIVERSIDADE ESTADUAL DE CAMPINAS  
INSTITUTO DE COMPUTAÇÃO

The contents of this report are the sole responsibility of the authors.  
O conteúdo deste relatório é de única responsabilidade dos autores.

# Extending unlocked matchings in graphs\*

A. A. Pereira<sup>†</sup>

C. N. Campos<sup>‡</sup>

## Abstract

Let  $M$  be a matching of a graph  $G$  and let  $S(M)$  and  $U(M)$  be the sets of  $M$ -saturated and  $M$ -unsaturated vertices of  $G$ , respectively. A vertex  $v \in U(M)$  is unlocked if there exists a vertex  $u \in N(v) \cap U(M)$ . The matching  $M$  is unlocked if every vertex in  $U(M)$  is unlocked. We say that a graph  $G$  is  $k$ -fully-extendable if every unlocked matching  $M$  in  $G$  of size  $k$  can be extended to a perfect matching of  $G$ . In this work, we study  $k$ -fully-extendable graphs of order  $2n$ , characterizing the cases  $k = n - 1$  and  $k = n - 2$ . For the case  $k = n - 2$ , we provide additional necessary conditions, based on degree bounds, as well as sufficient conditions, based on classes of graphs.

## 1 Introduction

Let  $G$  be a finite, undirected and simple graph with vertex set  $V(G)$  and edge set  $E(G)$ . Let  $N(v) = \{u \in V(G) : uv \in E(G) \text{ and } u \neq v\}$  be the (*open*) *neighbourhood* of a vertex  $v \in V(G)$ . Each  $u \in N(v)$  is called a *neighbour* of  $v$ . The *degree* of a vertex  $v \in V(G)$  is  $d_G(v) = |N(v)|$ . The *minimum degree* of  $G$  is denoted  $\delta(G)$  and the *maximum degree* is denoted  $\Delta(G)$ . Given  $S \subseteq V(G)$ , we denote  $\bar{S} = V(G) \setminus S$ . For a subgraph  $H$  of  $G$ , we analogously define  $\bar{H} \cong G[V(G) \setminus V(H)]$ . A *matching*  $M$  in  $G$  is a set of pairwise non-adjacent edges. A vertex  $v \in V(G)$  is said to be  *$M$ -saturated* if it is incident with some edge in  $M$ ; otherwise,  $v$  is said to be  *$M$ -unsaturated*. Denote  $S(M)$  and  $U(M)$  as the sets of  $M$ -saturated and  $M$ -unsaturated vertices of  $G$ , respectively. Set  $M$  is a *perfect matching* of a graph  $G$  when  $S(M) = V(G)$ . A graph is *matchable* if it admits a perfect matching. We say that  $M$  *extends* to a perfect matching if  $M$  is a subset of some perfect matching of  $G$ .

Matching theory has long been a focal point of Graph Theory, due both to its significant structural results [8, 10, 14, 19, 32] and its wide-ranging applications [9, 30, 35, 36]. As the theory developed, researchers investigated conditions under which matchings extend to perfect matchings. In this direction, Plummer [23] introduced the concept of  $k$ -extendable graphs in 1980.

We say that a graph  $G$  is  *$k$ -extendable* if every matching  $M$  of  $G$  with  $|M| = k$  can be extended to a perfect matching of  $G$ . In the special case  $k = 1$ ,  $k$ -extendability has been widely studied in the literature [15, 20] under the concept of a *matching-covered graph* – a graph in which every edge lies in some perfect matching. The problem has also been studied for other fixed values of  $k$ .

For  $k = n - 1$ , Anunchuen and Caccetta [3] proved that a matchable graph of order  $2n$  is  $(n - 1)$ -extendable if and only if it is isomorphic to  $K_{2n}$  or  $K_{n,n}$ . In a subsequent work [4], the same authors provided a complete characterization for  $k = n - 2$ . They showed that a matchable graph  $G$  of order  $2n$ , with  $n \geq 5$ , is  $(n - 2)$ -extendable if and only if: either  $G \cong K_{n,n}$  or  $G \cong K_{2n}$ ;

---

\*This research was supported by CAPES, grants 88887.520851/2020-00 and 88887.600764/2021-00, and by DAAD, grant 91882084.

<sup>†</sup>Instituto de Computação, UNICAMP, 13083-852 Campinas, SP. [alessandra.pereira@ic.unicamp.br](mailto:alessandra.pereira@ic.unicamp.br)

<sup>‡</sup>Instituto de Computação, UNICAMP, 13083-852 Campinas, SP. [cnc@unicamp.br](mailto:cnc@unicamp.br)

or  $G$  is bipartite with minimum degree  $n - 1$ ; or  $G$  has minimum degree  $2n - 3$  and contains a maximum independent set of order at most two; or, finally,  $G$  has minimum degree  $2n - 2$ .

The theory of  $k$ -extendable graphs has been explored in many directions, such as the establishment of necessary and sufficient conditions based on forbidden subgraphs [25], the analysis of criticality [2], the study of relationships with other graph parameters [6, 18, 21] and the investigation of their behaviour under graph operations [5, 11]. In addition, since Hackfeld and Koster [12] proved that the original problem is co-NP-complete, it has also been studied in classes of graphs, including bipartite, regular and planar graphs [1, 16, 24]. These and other related results are presented in three surveys by Plummer [26, 27, 28].

A key obstacle to extending a matching arises from the neighbourhood of an unsaturated vertex. In particular, for a non-perfect matching  $M$  of  $G$ , if there exists  $u \in U(M)$  such that  $N(u) \subseteq S(M)$ , then  $M$  is not extendable to a perfect matching of  $G$ . Motivated by this property, we analyse extendable matchings without this type of neighbourhood obstruction.

Let  $M$  be a matching of a graph  $G$ . We say that a vertex  $v \in U(M)$  is *unlocked* if there exists a vertex  $u \in N(v) \cap U(M)$ . We extend this notion to matchings by saying that  $M$  is *unlocked* if every vertex in  $U(M)$  is unlocked. Moreover, a graph  $G$  is  *$k$ -fully-extendable* if every unlocked matching  $M$  of  $G$  with  $|M| = k$  can be extended to a perfect matching of  $G$ . Observe that every matchable graph of order  $2n$  is trivially 0-fully-extendable and  $n$ -fully-extendable. For a given graph  $G$ , we define  $\mathcal{M}_k^f = \{M : M \text{ is an unlocked matching of } G \text{ and } |M| = k\}$ .

Using a different terminology, in 1997, Limaye and Sarvate [17] proved that every hypercube  $Q_n$ ,  $n \geq 2$ , is  $k$ -fully-extendable,  $1 \leq k \leq n$ . Furthermore, the authors showed that, for  $Q_4$ , there exists an unlocked matching of size 5 (i.e.  $n + 1$ ) that cannot be extended to a perfect matching, thereby confirming the tightness of the bound. For  $n \geq 5$ , Vandebussche and West [33] strengthened these results by showing that the hypercube  $Q_n$  is  $k$ -fully-extendable for  $1 \leq k \leq 2n - 4$ . Recently [22], we proved that the cartesian product  $Q_n \square P_m$ ,  $n \geq 4$  and  $m \geq 2$ , is  $k$ -fully-extendable,  $1 \leq k \leq 2n - 2$ .

One can note that, if a graph is  $k$ -extendable, then every matching  $M$  of  $G$  with  $|M| = k$  is unlocked. Therefore, a  $k$ -extendable graph is also  $k$ -fully-extendable. However, the converse is not necessarily true. For instance, every hypercube  $Q_n$ ,  $n \geq 3$ , is  $n$ -fully-extendable but not  $n$ -extendable. In fact, for  $n \geq 3$ , there exists a matching  $M$  of  $Q_n$  with  $|M| = n$  such that, for some  $u \in U(M)$ ,  $N(u) \subseteq M$ . Consequently,  $M$  is not extendable.

In this work, we study the extendability of unlocked matchings by investigating the concept of  $k$ -fully-extendable graphs of order  $2n$  for  $k \in \{n - 1, n - 2\}$ . First, we prove that every matchable graph of order  $2n$  is  $(n - 1)$ -fully-extendable. Then, we address the case of  $(n - 2)$ -fully-extendable graphs, for which we provide a characterization in terms of forbidden subgraphs and derive additional necessary and sufficient conditions involving classes of graphs and degree bounds.

## 2 Preliminaries

In this section, we present some results on matchable and hamiltonian graphs used to establish our main results. We begin with the classical theorem from Tutte [32]. Let  $G$  be a graph and  $S \subseteq V(G)$ . Denote  $o(G \setminus S)$  the number of connected components of  $G \setminus S$  having odd order.

**Theorem 1** (Tutte [32]). *A graph  $G$  is matchable if and only if  $o(G \setminus S) \leq |S|$  for every  $S \subseteq V(G)$ .*  $\square$

A *claw* is a graph isomorphic to  $K_{1,3}$ . Let  $H$  be a claw. We call the vertex  $c \in V(H)$  with  $d_H(c) = 3$  the *central vertex* of  $H$ . An *induced claw* of a graph  $G$  is an induced subgraph of  $G$  isomorphic to a claw. A graph  $G$  is *claw-free* if it does not contain induced claws. The following theorem states that every connected claw-free graph of even order is matchable.

**Theorem 2** (Sumner [31], Las Vergnas [34]). *Let  $G$  be a connected graph of even order. If  $G$  is claw-free, then  $G$  is matchable.*  $\square$

We conclude this section by presenting a theorem that unifies some sufficient conditions for hamiltonian graphs. For a graph  $G$ , let the *connectivity*  $\kappa(G)$  of  $G$  be the maximum value of  $k$  for which  $G$  is  $k$ -connected.

**Theorem 3** (Dirac [7], Pósa [29], Häggkvist and Nicoghossian [13]). *Let  $G$  be a graph of order  $n \geq 3$ . Then,  $G$  is hamiltonian if any of the following holds:*

- (i)  $\delta(G) \geq \frac{n}{2}$ ;
- (ii) for every integer  $1 \leq k < \frac{n}{2}$ , the number of vertices of degree at most  $k$  is strictly less than  $k$ ;
- (iii)  $G$  is 2-connected and  $\delta(G) \geq \frac{n+\kappa(G)}{3}$ .  $\square$

### 3 Results

Now, we present our results on fully-extendable graphs. First, Proposition 4 states that a matchable graph contains unlocked matchings of every possible size.

**Proposition 4.** *Let  $G$  be a matchable graph of order  $2n$ ,  $n \geq 1$ . Then, for every  $0 \leq k \leq n$ , there exists an unlocked matching of  $G$  with  $|M| = k$ .*  $\square$

In Theorem 5, our first result, we show that  $G$  is also  $(n - 1)$ -fully-extendable. This result is an immediate consequence of Proposition 4.

**Theorem 5.** *Every matchable graph  $G$  of order  $2n$ ,  $n \geq 1$ , is  $(n - 1)$ -fully-extendable.*  $\square$

Our primary focus in this work is the study of  $(n - 2)$ -fully-extendability in graphs. This case is more subtle and requires additional care, in contrast to the relative straightforwardness of the  $n - 1$  case discussed in Theorem 5.

In Theorem 6, we establish a characterization of  $(n - 2)$ -fully-extendable graphs in terms of induced claws, which provides the foundation for the subsequent results. We then strengthen this characterization in Theorems 7 and 8, by deriving further necessary and sufficient conditions based on classes of graphs and minimum degree bounds.

**Theorem 6.** *Let  $G$  be a matchable graph of order  $2n$ ,  $n \geq 2$ . Then,  $G$  is  $(n - 2)$ -fully-extendable if and only if for every induced claw  $H$  of  $G$ ,  $\overline{H}$  is non-matchable.*  $\square$

**Theorem 7.** *Let  $G$  be a matchable graph of order  $2n$ ,  $n \geq 2$ . If  $G$  is claw-free or bipartite, then  $G$  is  $(n - 2)$ -fully-extendable.*  $\square$

**Theorem 8.** *Let  $G$  be a connected graph of order  $2n$ ,  $n \geq 4$ . If  $G$  is  $(n - 2)$ -fully-extendable and  $G$  is non-claw-free, then  $\delta(G) \leq n$ .*

*Proof.* Let  $G$  be a connected graph of order  $2n$ ,  $n \geq 4$ . Suppose that  $G$  is  $(n - 2)$ -fully-extendable. Suppose also that  $G$  is non-claw-free. Let  $\mathcal{H} = \{H : H \text{ is an induced claw of } G\}$ . Note that, for every  $H \in \mathcal{H}$ , we have  $\delta(\overline{H}) \geq \delta(G) - 4$  and  $|V(\overline{H})| = 2n - 4$ .

Suppose  $\delta(G) \geq n + 2$ . Let  $H \in \mathcal{H}$ . Then,  $\delta(\overline{H}) \geq n - 2 \geq \frac{|V(\overline{H})|}{2}$ . Applying Theorem 3,  $\overline{H}$  is hamiltonian and, hence, matchable. By Theorem 6, we conclude that  $G$  is not  $(n - 2)$ -fully-extendable.

Suppose  $\delta(G) = n + 1$ . If there exists  $H \in \mathcal{H}$  such that  $\delta(\overline{H}) \geq n - 2$ , then, by the previous reasoning,  $G$  is not  $(n - 2)$ -fully-extendable. Hence, we may assume that, for every  $H \in \mathcal{H}$ ,  $\delta(\overline{H}) = n - 3$ .

Let  $H \in \mathcal{H}$  and  $V_\delta = \{v \in V(\overline{H}) : d_{\overline{H}}(v) = n - 3\}$ . If  $|V_\delta| \leq n - 4$ , then, by Theorem 3,  $\overline{H}$  is hamiltonian, which implies that  $G$  is not  $(n - 2)$ -fully-extendable. We conclude that  $|V_\delta| \geq n - 3$ . Let  $u \in V(H)$  be the central vertex of  $H$  and  $V(H) \setminus \{u\} = \{v_1, v_2, v_3\}$ . Note that every  $w \in V_\delta$  is adjacent to all vertices of  $H$  and  $G[\{w, v_1, v_2, v_3\}] \cong K_{1,3}$ . Thus, we can choose  $H \in \mathcal{H}$  so that  $d_G(u) = n + 1$ . This implies  $|V_\delta| \leq n - 2$ .

Suppose  $\kappa(\overline{H}) = 0$ . Let  $G_1$  and  $G_2$  denote two distinct connected components of  $\overline{H}$ . Since  $|V(\overline{H})| = 2n - 4$ , we may assume  $|V(G_1)| \leq n - 2$ . Moreover, considering that  $\delta(\overline{H}) = n - 3$ , it follows that  $|V(G_1)| \geq n - 2$  and  $|V(G_2)| \geq n - 2$ . Hence,  $|V(G_1)| = |V(G_2)| = n - 2$ . Thus, each  $G_i$ ,  $i \in \{1, 2\}$ , contains exactly  $n - 2$  vertices of degree  $n - 3$  in  $\overline{H}$ . Furthermore, since  $\delta(G) \geq n + 1$ , each vertex of  $G_i$  is adjacent in  $G$  to every vertex of  $H$ . This implies  $|V_\delta| = 2n - 4$ , contradicting  $|V_\delta| \leq n - 2$ . Therefore,  $\kappa(\overline{H}) \geq 1$ .

Suppose  $\kappa(\overline{H}) = 1$ . Then,  $\overline{H}$  has a cut vertex  $x$ . Let  $G'_1$  and  $G'_2$  denote two distinct connected components of  $G' \cong \overline{H}[V(\overline{H}) \setminus \{x\}]$ . Without loss of generality, assume that  $|V(G'_1)| \leq |V(G'_2)|$ . Note that  $|V(G')| = 2n - 5$  and  $\delta(G') \in \{n - 3, n - 4\}$ . If  $\delta(G') = n - 3$ , then, since  $n \geq 4$ , we would have  $|V(G'_1)| \geq n - 2$  and  $|V(G'_2)| \geq n - 2$ , implying  $|V(G')| \geq 2n - 4$ , a contradiction. Hence,  $\delta(G') = n - 4$ .

First, consider  $n = 4$ . In this case,  $\delta(G') = 0$  and  $\overline{H}$  must be isomorphic to one of the two graphs shown in Fig. 1. Since  $n - 3 \leq |V_\delta| \leq n - 2$ , it follows that  $\overline{H}$  can only be isomorphic to the graph in Fig. 1(a). Thus  $\overline{H}$  is matchable, and by Theorem 6,  $G$  is not  $(n - 2)$ -fully-extendable.



Figure 1: Graph  $\overline{H}$  for  $n = 4$  when  $\delta(G') = n - 4$ .

Now suppose  $n \geq 5$ . Recall that  $|V(G')| = 2n - 5$ . Therefore, we may assume  $|V(G'_1)| \leq n - 3$ . Since  $\delta(G') = n - 4$ , every vertex of  $G'_1$  has at least  $n - 4$  neighbours in  $G'_1$  and it follow that  $|V(G'_1)| \geq (n - 4) + 1 = n - 3$ . Hence,  $|V(G'_1)| = n - 3$ . This implies  $|V(G'_2)| = n - 2$ . Note that  $G'_1 \cong K_{n-3}$ . Suppose  $\delta(G'_2) = n - 3$ . Then, similarly to the previous case, we conclude that  $G'_2 \cong K_{n-2}$ . Thus,  $\overline{H}$  is a connected claw-free graph. By Theorem 2,  $\overline{H}$  is matchable and, hence,  $G$  is not  $(n - 2)$ -fully-extendable. Therefore, we can assume  $\delta(G'_2) = n - 4$ .

Suppose, first,  $n = 5$ . Recalling that  $|V(G'_1)| = 2$  and  $|V(G'_2)| = 3$ , we conclude that  $G'_1 \cong K_2$  and  $G'_2 \cong P_3$ . If  $y$  is a vertex of degree one in  $G'_2$ , then  $y$  is adjacent to  $x$ ; otherwise  $d_G(y) \leq 5 < n + 1 = 6$ . Then,  $\overline{H}$  contains a spanning path  $P$  of six vertices consisting of the vertices of  $G'_2$ , followed by  $x$ , and then the vertices of  $G'_1$ . Since  $P$  is matchable, by Theorem 7,  $G$  is not  $(n - 2)$ -fully-extendable.

Suppose  $n \geq 6$ . Then,  $|V(G'_2)| \geq 3$  and  $n - 4 \geq \frac{n-2}{2}$ . Therefore, by Theorem 3,  $G'_2$  is hamiltonian. Considering that  $|V(G')| = 2n - 5$ ,  $|V(G'_1)|$  and  $|V(G'_2)|$  have opposite parity. Let  $x_1 \in V(G'_1)$  and  $x_2 \in V(G'_2)$  be neighbours of  $x$ . Suppose  $|V(G'_1)|$  even. Then, since  $G'_1 \cong K_{n-3}$ ,  $G'_1$  has a perfect matching  $M_1$ . Moreover, since  $|V(G'_2)|$  is odd and  $G'_2$  is hamiltonian,  $G'_2[V(G'_2) \setminus \{x_2\}]$  has a perfect matching  $M_2$ . We conclude that  $M_1 \cup M_2 \cup \{xx_2\}$  is a perfect matching of  $\overline{H}$  and  $G$

is not  $(n-2)$ -fully-extendable. The case  $|V(G'_1)|$  odd and  $|V(G'_2)|$  even is analogous.

Finally, suppose  $\kappa(\overline{H}) \geq 2$ . If  $\kappa(\overline{H}) \leq n-5$ , then  $\frac{2n-4+\kappa(\overline{H})}{3} \leq n-3$ . Hence, by Theorem 3,  $\overline{H}$  is hamiltonian since  $\delta(\overline{H}) = n-3$ , and we are done again. Suppose, now,  $\kappa(\overline{H}) \geq n-4$ . If  $\overline{H}$  is non-matchable, then, by Theorem 1, there exists  $X \subseteq V(\overline{H})$  such that for  $G' \cong \overline{H}[V(\overline{H}) \setminus X]$ ,  $o(G') > |X|$ . It follows that  $|X| \leq n-3$  since  $|V(\overline{H})| = 2n-4$ . On the other hand,  $|X| \geq n-4$  since  $\kappa(\overline{H}) \geq n-4$ . We conclude that  $n-4 \leq |X| \leq n-3$ .

Suppose  $|X| = n-3$ . Since  $o(G') > |X|$ , we conclude that  $o(G') \geq n-2$ . On the other hand,  $|V(G')| = |V(\overline{H})| - |X| = n-1$ . Hence,  $o(G') \leq n-1$ . If  $o(G') = n-2$ , then  $|V(G')| - o(G') = 1$ , a contradiction since a connected component of even order requires at least two vertices. Therefore,  $o(G') = n-1$  and  $G'$  consists of  $n-1$  isolated vertices. The equalities  $|X| = n-3$  and  $\delta(\overline{H}) = n-3$  imply that every vertex of  $G'$  has degree  $n-3$  in  $\overline{H}$ , contradicting the fact that  $|V_\delta| \leq n-2$ .

Suppose  $|X| = n-4$ . Then, since  $\overline{H}$  is 2-connected,  $|X| \geq 2$  and, thus,  $n \geq 6$ . Moreover,  $n-3 \leq o(G') \leq n$ . If  $G'$  has an isolated vertex  $z$ , then  $d_{\overline{H}}(z) \leq n-4$ , a contradiction. Therefore, for every connected component  $G'_i$  of  $G'$  with odd order, we have  $|V(G'_i)| \geq 3$ . Hence,  $o(G') \leq \frac{n}{3}$ . However,  $\frac{n}{3} \geq o(G') > |X| = n-4$  only when  $n < 6$ , another contradiction.

We conclude that if  $G$  is  $(n-2)$ -fully-extendable and  $G$  is non-claw-free, then  $\delta(G) \leq n$ .  $\square$

It is well known that connected bipartite matchable graphs of order  $2n$  have  $\delta(G) \leq n$ . Moreover, equality is attained by the complete bipartite graphs. By Theorem 7, these graphs are  $(n-2)$ -fully-extendable and, from the perspective of Theorem 8, they demonstrate that the bound stated therein is tight. Next, we show that the bound remains tight even when considering non-bipartite graphs.

**Theorem 9.** *For every integer  $n \geq 5$ , there exists a matchable, non-bipartite, and non-claw-free graph  $G$  of order  $2n$  with  $\delta(G) = n$  that is  $(n-2)$ -fully-extendable.*

*Proof.* Let  $n$  be an integer such that  $n \geq 5$ . The proof is constructive and divided into cases depending on the parity of  $n$ .

**Case 1.**  $n$  even.

Let  $G_1$ ,  $G_2$ , and  $G_3$  be three pairwise disjoint graphs such that  $G_1 \cong K_{n-2}$ ,  $G_2 \cong K_{n-1}$ ,  $V(G_3) = \{x, y, z\}$  and  $E(G_3) = \emptyset$ . We construct  $G$  as follows:

- $V(G) = V(G_1) \cup V(G_2) \cup V(G_3)$ ;
- $E(G) = E(G_1) \cup E(G_3) \cup E_{13} \cup E_{23}$  such that:
  - $E_{13} = \{ux, uy, uz : u \in V(G_1)\}$ ;
  - given a bipartition  $\{V_{xy}, V_{yz}\}$  of  $V(G_2)$  such that  $|V_{xy}| = \lfloor \frac{n-1}{2} \rfloor$  and  $|V_{yz}| = \lceil \frac{n-1}{2} \rceil$ , we define  $E_{23} = \{vx, vy : v \in V_{xy}\} \cup \{vy, vz : v \in V_{yz}\}$ .

This construction is illustrated for  $n = 6$  in Fig. 2. Next, we show that  $\delta(G) = n$  and that  $G$  is a non-bipartite, non-claw-free, matchable, and  $(n-2)$ -fully-extendable graph.

First, note that  $|V(G)| = (n-2) + 3 + (n-1) = 2n$ . For  $u \in V(G_1)$ , it follows that  $d_G(u) = (n-3) + 3 = n$  and for  $u \in V(G_2)$ ,  $d_G(u) = (n-2) + 2 = n$ . For the vertices of  $G_3$ ,  $d_G(y) = (n-2) + (n-1) = 2n-3 \geq n$ ,  $d_G(x) = (n-2) + \lfloor \frac{n-1}{2} \rfloor$  and  $d_G(z) = (n-2) + \lceil \frac{n-1}{2} \rceil$ . Since  $n$  is even and  $n \geq 6$ , we have  $d_G(x) \geq n$  and  $d_G(z) \geq n$ . We conclude that  $\delta(G) = n$ .

Since  $G_1 \cong K_{n-2}$ ,  $G$  is non-bipartite. Moreover, for every  $u \in V(G_1)$ ,  $G[\{u, x, y, z\}] \cong K_{1,3}$ . We now prove that  $G$  is matchable. As  $G_1 \cong K_{n-2}$  and  $n-2$  is even,  $G_1$  has a perfect matching

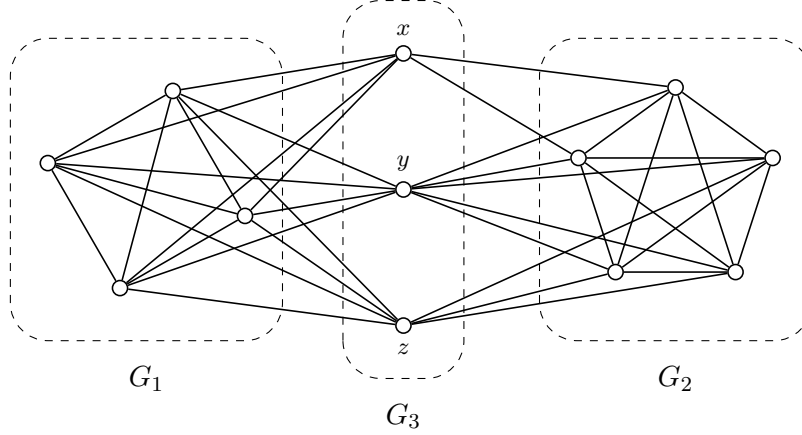


Figure 2: Construction of a matchable, non-bipartite and non-claw-free graph  $G$  of order  $2n = 12$  with  $\delta(G) = n = 6$  that is  $(n - 2)$ -fully-extendable.

$M_1$ . Let  $\{u_x, u_y, u_z\} \subseteq V(G_2)$  be a set of distinct vertices such that  $\{u_x x, u_y y, u_z z\} \subseteq E(G)$ . Then,  $G[V(G_2) \setminus \{u_x, u_y, u_z\}] \cong K_{n-4}$  has a perfect matching  $M_3$ . Thus,  $M = M_1 \cup M_3 \cup \{u_x x, u_y y, u_z z\}$  is a perfect matching of  $G$ .

It remains to show that  $G$  is  $(n - 2)$ -fully-extendable. Let  $H$  be an induced claw of  $G$  such that  $V(H) = \{c, v_1, v_2, v_3\}$  with  $c$  being its central vertex. Since  $G_1 \cong K_{n-2}$  and  $G_2 \cong K_{n-1}$ , we have  $|\{v_1, v_2, v_3\} \cap V(G_1)| \leq 1$  and  $|\{v_1, v_2, v_3\} \cap V(G_2)| \leq 1$ . Furthermore, since  $y$  is adjacent to all vertices of  $G_2$  and every vertex of  $G_2$  is adjacent to exactly two vertices of  $G_3$ , it follows that  $c \notin V(G_2)$ . Also, note that  $E(G_3) = \emptyset$ . Therefore,  $c \notin V(G_3)$ . Thus, we conclude that  $c \in V(G_1)$ . Consequently,  $\bar{H}$  consists of two connected components isomorphic to  $K_{n-3}$  and  $K_{n-1}$ . Since  $n$  is even,  $K_{n-3}$  and  $K_{n-1}$  are non-matchable. Therefore, by Theorem 6,  $G$  is  $(n - 2)$ -fully-extendable.

**Case 2.  $n$  odd.**

Let  $G_1, G_2$ , and  $G_3$  be three pairwise disjoint graphs such that  $G_1 \cong K_{n-2}$ ,  $G_2 \cong K_{n-2}$ ,  $V(G_3) = \{w, x, y, z\}$  and  $E(G_3) = \{wx, wy, wz\}$ . We construct  $G$  as follows:

- $V(G) = V(G_1) \cup V(G_2) \cup V(G_3)$ ;
- $E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup E_w \cup E_x \cup E_y \cup E_z$ , such that, for fixed vertices  $v_1 \in V(G_1)$  and  $v_2 \in V(G_2)$ :
  - $E_w = \{vw : v \in V(G_1) \cup V(G_2)\}$ ;
  - $E_x = \{vx : v \in V(G_1) \cup \{v_2\}\}$ ;
  - $E_y = \{vy : v \in V(G_2) \cup \{v_1\}\}$ ;
  - $E_z = \{vz : v \in V(G_1) \cup V(G_2) \setminus \{v_1, v_2\}\}$ .

This construction is illustrated for  $n = 7$  in Fig. 3. Next, we show that  $\delta(G) = n$  and that  $G$  is a non-bipartite, non-claw-free, matchable, and  $(n - 2)$ -fully-extendable graph.

First, observe that  $|V(G)| = (n - 2) + 4 + (n - 2) = 2n$ . For  $u \in V(G_1) \setminus \{v_1\}$ , it follows that  $d_G(u) = (n - 3) + 1 + 1 + 1 = n$  and  $d_G(v_1) = (n - 3) + 1 + 1 + 1 = n$ . Similarly, for  $u \in V(G_2) \setminus \{v_2\}$ ,  $d_G(u) = (n - 3) + 1 + 1 + 1 = n$  and  $d_G(v_2) = (n - 3) + 1 + 1 + 1 = n$ . For the vertices of  $G_3$ ,  $d_G(w) = (n - 2) + (n - 2) + 3 = 2n - 1 \geq n$ ,  $d_G(x) = d_G(y) = (n - 2) + 1 + 1 = n$  and  $d_G(z) = (n - 3) + (n - 3) + 1 = 2n - 5$ . Since  $n \geq 5$ , we have  $d_G(z) \geq n$ . Hence,  $\delta(G) = n$ .

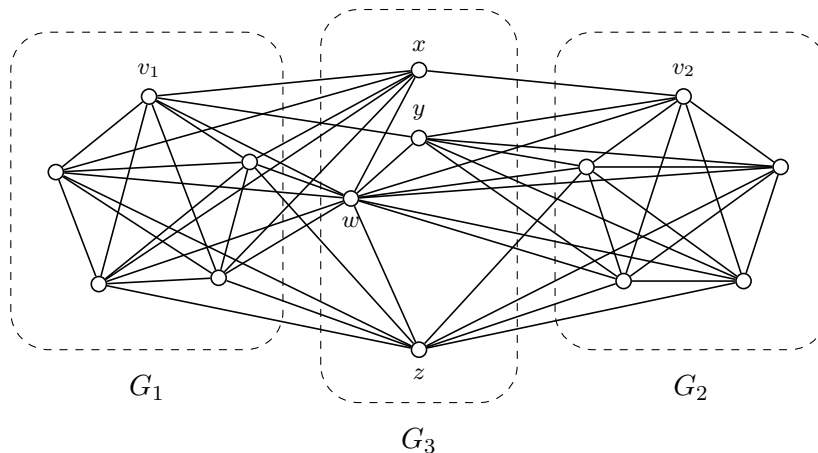


Figure 3: Construction of a matchable, non-bipartite and non-claw-free graph  $G$  of order  $2n = 14$  with  $\delta(G) = n = 7$  that is  $(n - 2)$ -fully-extendable.

Once again, since  $G_1 \cong K_{n-2}$ ,  $G$  is non-bipartite. Moreover,  $G$  contains an induced claw since  $G_2 \cong K_{1,3}$ . In order to prove that  $G$  is matchable, note that  $G[V(G_1) \setminus \{v_1\}] \cong K_{n-3}$  and  $G[V(G_2) \setminus \{v_2\}] \cong K_{n-3}$  have perfect matchings  $M_1$  and  $M_2$ , respectively. Thus, we conclude that  $M = M_1 \cup M_2 \cup \{v_1y, v_2x, wz\}$  is a perfect matching of  $G$ .

It remains to show that  $G$  is  $(n - 2)$ -fully-extendable. Let  $H$  be an induced claw of  $G$  such that  $V(H) = \{c, t_1, t_2, t_3\}$  with  $c$  being its central vertex. Since  $G_1 \cong K_{n-2}$  and  $G_2 \cong K_{n-2}$ , we have  $|\{t_1, t_2, t_3\} \cap V(G_1)| \leq 1$  and  $|\{t_1, t_2, t_3\} \cap V(G_2)| \leq 1$ . Suppose  $c \in V(G_1)$ . Then,  $\{t_1, t_2, t_3\} \subseteq V(G_1) \cup V(G_3)$ . By construction, every vertex of  $V(G_1) \setminus \{v_1\}$  is adjacent to  $w, x$  and  $z$ , and  $v_1$  is adjacent to  $w, x$  and  $y$ . Therefore, since  $w$  is also adjacent to  $x, y$  and  $z$ , it follows that  $|\{t_1, t_2, t_3\} \cap V(G_3)| \leq 2$ . This implies  $|\{t_1, t_2, t_3\} \cap V(G_1)| = 1$ ,  $|\{t_1, t_2, t_3\} \cap V(G_3)| = 2$  and  $w \notin \{t_1, t_2, t_3\}$ . If  $c = v_1$ , then we can assume  $\{t_1, t_2\} = \{x, y\}$  and  $t_3 \in V(G_1) \setminus \{v_1\}$ . If  $c \in V(G_1) \setminus \{v_1\}$ , then we can assume  $\{t_1, t_2\} = \{x, z\}$  and  $t_3 \in V(G_1) \setminus \{c\}$ . However, this is a contradiction since in both cases  $x \in \{t_1, t_2\}$  and  $x$  is adjacent to every vertex of  $G_1$ , including  $t_3$ . Therefore,  $c \notin V(G_1)$ . Analogously,  $c \notin V(G_2)$ . We conclude that  $c \in V(G_3)$ . Considering that  $w$  is adjacent to all the other vertices of  $G$  and  $|\{t_1, t_2, t_3\} \cap V(G_1)| \leq 1$  and  $|\{t_1, t_2, t_3\} \cap V(G_2)| \leq 1$ , it follows that  $c = w$ . Consequently, either  $V(H) = \{w, x, y, z\}$  or  $V(H) = \{w, v_1, v_3, z\}$ . In both cases,  $\bar{H}$  consists of two connected components isomorphic to  $K_{n-2}$ . Therefore, again by Theorem 6,  $G$  is  $(n - 2)$ -fully-extendable.  $\square$

## 4 Concluding remarks

The problem of  $k$ -fully-extendability has previously appeared in the literature as the study of extendability of matchings of size  $k$  that do not saturate the neighbourhood of any unsaturated vertex. Motivated by this, we introduce the new terminology of  $k$ -fully-extendable graphs and investigate their structural properties.

In particular, we have shown, in Theorem 5, that every matchable graph of order  $2n$  is  $(n - 1)$ -fully-extendable. In Theorem 6, we present a general characterization of  $(n - 2)$ -fully-extendable graphs in terms of induced claws. Building on this characterization, Theorem 7 shows that claw-free and bipartite graphs are  $(n - 2)$ -fully-extendable. In Theorem 8, we prove that a graph that is  $(n - 2)$ -fully-extendable and non-claw-free must satisfy  $\delta(G) \leq n$ . Moreover, in Theorem 9, we prove that this bound is tight even when considering non-bipartite graphs.

In order to further characterize  $(n - 2)$ -fully-extendable graphs using Theorems 7 and 8, we need to identify the conditions under which matchable, non-bipartite, non-claw-free graphs of order  $2n$  with  $\delta(G) \leq n$  are  $(n - 2)$ -fully-extendable. Another challenging open problem is to study  $k$ -fully-extendable graphs for  $1 \leq k \leq n - 3$ . Moreover, it is also interesting to explore extremal aspects by characterizing maximal and minimal  $k$ -fully-extendable graphs, as well as to investigate the relationship with other parameters and their behaviour under graph operations.

## References

- [1] R. Aldred, D. A Holton, and D. Lou.  $N$ -extendability of symmetric graphs. *Journal of graph theory*, 17(2):253–261, 1993.
- [2] R. Aldred and M. Plummer. Extendability and criticality in matching theory. *Graphs and combinatorics*, 36(3):573–589, 2020.
- [3] N. Anunchuen and L. Caccetta. On minimally  $k$ -extendable graphs. *Australasian Journal of Combinatorics*, 9:153–168, 1994.
- [4] N Anunchuen and L Caccetta. On  $(n-2)$ -extendable graphs. *J. Combin. Math. Combin. Comput*, 16:115–128, 1994.
- [5] B. Bai, Z. Wu, X. Yang, and Q. Yu. Lexicographic product of extendable graphs. *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, 33(2):197–204, 2010.
- [6] C. Chen. Binding number and toughness for matching extension. *Discrete Mathematics*, 146(1-3):303–306, 1995.
- [7] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.
- [8] J. Edmonds. Paths, trees, and flowers. *Canadian Journal of mathematics*, 17:449–467, 1965.
- [9] D. Gale and L. Shapley. College admissions and the stability of marriage. *The American mathematical monthly*, 69(1):9–15, 1962.
- [10] T. Gallai. Kritische graphen II. *Magyar Tud. Akad. Mat. Kutato Int. Kozl.*, 8:373–395, 1963.
- [11] E. Györi and M. Plummer. The Cartesian product of a  $k$ -extendable and an  $l$ -extendable graph is  $(k + l + 1)$ -extendable. *The Julius Petersen Graph Theory Centennial*, 6:417, 1992.
- [12] J. Hackfeld and A. Koster. The matching extension problem in general graphs is co-NP-complete. *Journal of Combinatorial Optimization*, 35:853–859, 2018.
- [13] R. Häggkvist and G.G. Nicoghossian. A remark on hamiltonian cycles. *Journal of Combinatorial Theory, Series B*, 30(1):118–120, 1981.
- [14] P. Hall. On representatives of subsets. *Classic Papers in Combinatorics*, pages 58–62, 1987.
- [15] J. He, E. Wei, D. Ye, and S. Zhai. On perfect matchings in matching covered graphs. *Journal of Graph Theory*, 90(4):535–546, 2019.
- [16] J. Lakhali and L. Litzler. A polynomial algorithm for the extendability problem in bipartite graphs. *Information processing letters*, 65(1):11–16, 1998.

- [17] Nirmala B Limaye and Dinesh G Sarvate. On  $r$ -extendability of the hypercube  $Q_n$ . *Mathematica Bohemica*, 122(3):249–255, 1997.
- [18] D. Lou. On matchability of graphs. *Australasian Journal of Combinatorics*, 21:201–210, 2000.
- [19] L. Lovász. On the structure of factorizable graphs. *Acta Mathematica Hungarica*, 23(1-2):179–195, 1972.
- [20] F. Lu, N. Kothari, X. Feng, and L. Zhang. Equivalence classes in matching covered graphs. *Discrete Mathematics*, 343(8):111945, 2020.
- [21] P. Maschlanka and L. Volkmann. Independence number in  $n$ -extendable graphs. *Discrete Mathematics*, 154(1-3):167–178, 1996.
- [22] A. A. Pereira and C. N. Campos. Matching Extendability in Cartesian Product of Hypercubes and Paths. Manuscript, 2025.
- [23] M. Plummer. On  $n$ -extendable graphs. *Discrete Mathematics*, 31(2):201–210, 1980.
- [24] M. Plummer. A theorem on matchings in the plane. *Ann. Discrete Math*, 41:347–354, 1988.
- [25] M. Plummer. Extending matchings in claw-free graphs. *Discrete mathematics*, 125(1-3):301–307, 1994.
- [26] M. Plummer. Extending matchings in graphs: a survey. *Discrete Mathematics*, 127(1-3):277–292, 1994.
- [27] M. Plummer. Extending matchings in graphs: an update. *Congr Numer*, 116:3, 1996.
- [28] M. Plummer. Recent progress in matching extension. *Building Bridges: Between Mathematics and Computer Science*, pages 427–454, 2008.
- [29] L. Pósa. A theorem concerning hamilton lines. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7:225–226, 1962.
- [30] A. Roth. *Online and matching-based market design*. Cambridge University Press, 2023.
- [31] D. P. Sumner. Graphs with 1-factors. *Proceedings of the American Mathematical Society*, 42(1):8–12, 1974.
- [32] W. Tutte. The factorization of linear graphs. *Journal of the London Mathematical Society*, 1(2):107–111, 1947.
- [33] J. Vandenbussche and D. B. West. Matching extendability in hypercubes. *SIAM Journal on Discrete Mathematics*, 23(3):1539–1547, 2009.
- [34] Las Vergnas. A note on matchings in graphs. *Cahiers Centre Études Recherche Opér.*, 17:257, 1975.
- [35] D. Vukičević. Applications of perfect matchings in chemistry. In *Structural Analysis of Complex Networks*, pages 463–482. Springer, 2010.
- [36] C. Witwer, I. Hofacker, and P. Stadler. Prediction of consensus rna secondary structures including pseudoknots. *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, 1(2):66–77, 2004.