

9

Relations

- 9.1 Relations and Their Properties
- 9.2 n -ary Relations and Their Applications
- 9.3 Representing Relations
- 9.4 Closures of Relations
- 9.5 Equivalence Relations
- 9.6 Partial Orderings

Relationships between elements of sets occur in many contexts. Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, a person and a relative, and so on. In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number and one that is larger than it, a real number x and the value $f(x)$ where f is a function, and so on. Relationships such as that between a program and a variable it uses, and that between a computer language and a valid statement in this language often arise in computer science.

Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets. Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.

In some computer languages, only the first 31 characters of the name of a variable matter. The relation consisting of ordered pairs of strings where the first string has the same initial 31 characters as the second string is an example of a special type of relation, known as an equivalence relation. Equivalence relations arise throughout mathematics and computer science. We will study equivalence relations, and other special types of relations, in this chapter.

9.1 Relations and Their Properties

Introduction



The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations. In this section we introduce the basic terminology used to describe binary relations. Later in this chapter we will use relations to solve problems involving communications networks, project scheduling, and identifying elements in sets with common properties.

DEFINITION 1

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B . We use the notation $a R b$ to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R , a is said to be **related to** b by R .

Binary relations represent relationships between the elements of two sets. We will introduce n -ary relations, which express relationships among elements of more than two sets, later in this chapter. We will omit the word *binary* when there is no danger of confusion.

Examples 1–3 illustrate the notion of a relation.

EXAMPLE 1

Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a, b) , where a is a student enrolled in course b . For instance, if Jason Goodfriend and Deborah Sherman are enrolled in CS518, the pairs

Proof: We first prove the “if” part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \dots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.




We will use mathematical induction to prove the only if part of the theorem. Note that this part of the theorem is trivially true for $n = 1$.

Assume that $R^n \subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R . To show this, assume that $(a, b) \in R^{n+1}$. Then, because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$. Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof. ◀

Exercises

- List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if
 - $a = b$.
 - $a + b = 4$.
 - $a > b$.
 - $a \mid b$.
 - $\gcd(a, b) = 1$.
 - $\text{lcm}(a, b) = 2$.
 - List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.
 - Display this relation graphically, as was done in Example 4.
 - Display this relation in tabular form, as was done in Example 4.
 - For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
 - $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
 - $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
 - $\{(2, 4), (4, 2)\}$
 - $\{(1, 2), (2, 3), (3, 4)\}$
 - $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
 - $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$
 - Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
 - a is taller than b .
 - a and b were born on the same day.
 - a has the same first name as b .
 - a and b have a common grandparent.
 - Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
 - everyone who has visited Web page a has also visited Web page b .
 - there are no common links found on both Web page a and Web page b .
 - there is at least one common link on Web page a and Web page b .
 - there is a Web page that includes links to both Web page a and Web page b .
 - Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
 - $x + y = 0$.
 - $x = \pm y$.
 - $x - y$ is a rational number.
 - $x = 2y$.
 - $xy \geq 0$.
 - $xy = 0$.
 - $x = 1$.
 - $x = 1$ or $y = 1$.
 - Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
 - $x \neq y$.
 - $xy \geq 1$.
 - $x = y + 1$ or $x = y - 1$.
 - $x \equiv y \pmod{7}$.
 - x is a multiple of y .
 - x and y are both negative or both nonnegative.
 - $x = y^2$.
 - $x \geq y^2$.
 - Show that the relation $R = \emptyset$ on a nonempty set S is symmetric and transitive, but not reflexive.
 - Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.
 - Give an example of a relation on a set that is
 - both symmetric and antisymmetric.
 - neither symmetric nor antisymmetric.
- A relation R on the set A is **irreflexive** if for every $a \in A$, $(a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.
- Which relations in Exercise 3 are irreflexive?
 - Which relations in Exercise 4 are irreflexive?
 - Which relations in Exercise 5 are irreflexive?
 - Which relations in Exercise 6 are irreflexive?
 - Can a relation on a set be neither reflexive nor irreflexive?
 - Use quantifiers to express what it means for a relation to be irreflexive.
 - Give an example of an irreflexive relation on the set of all people.

A relation R is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$. Exercises 18–24 explore the notion of an asymmetric relation. Exercise 22 focuses on the difference between asymmetry and antisymmetry.

18. Which relations in Exercise 3 are asymmetric?
 19. Which relations in Exercise 4 are asymmetric?
 20. Which relations in Exercise 5 are asymmetric?
 21. Which relations in Exercise 6 are asymmetric?
 22. Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answers.
 23. Use quantifiers to express what it means for a relation to be asymmetric.
 24. Give an example of an asymmetric relation on the set of all people.
 25. How many different relations are there from a set with m elements to a set with n elements?
-  Let R be a relation from a set A to a set B . The **inverse relation** from B to A , denoted by R^{-1} , is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** \bar{R} is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.
26. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find
 - a) R^{-1} .
 - b) \bar{R} .
 27. Let R be the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set of positive integers. Find
 - a) R^{-1} .
 - b) \bar{R} .
 28. Let R be the relation on the set of all states in the United States consisting of pairs (a, b) where state a borders state b . Find
 - a) R^{-1} .
 - b) \bar{R} .
 29. Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f . That is, $R = \{(a, f(a)) \mid a \in A\}$. What is the inverse relation R^{-1} ?
 30. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find
 - a) $R_1 \cup R_2$.
 - b) $R_1 \cap R_2$.
 - c) $R_1 - R_2$.
 - d) $R_2 - R_1$.
 31. Let A be the set of students at your school and B the set of books in the school library. Let R_1 and R_2 be the relations consisting of all ordered pairs (a, b) , where student a is required to read book b in a course, and where student a has read book b , respectively. Describe the ordered pairs in each of these relations.
 - a) $R_1 \cup R_2$
 - b) $R_1 \cap R_2$
 - c) $R_1 \oplus R_2$
 - d) $R_1 - R_2$
 - e) $R_2 - R_1$
 32. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

33. Let R be the relation on the set of people consisting of pairs (a, b) , where a is a parent of b . Let S be the relation on the set of people consisting of pairs (a, b) , where a and b are siblings (brothers or sisters). What are $S \circ R$ and $R \circ S$?

Exercises 34–37 deal with these relations on the set of real numbers:

$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the “greater than” relation,

$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the “greater than or equal to” relation,

$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the “less than” relation,

$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the “less than or equal to” relation,

$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the “equal to” relation,

$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the “unequal to” relation.

34. Find

- | | |
|-----------------------|-----------------------|
| a) $R_1 \cup R_3$. | b) $R_1 \cup R_5$. |
| c) $R_2 \cap R_4$. | d) $R_3 \cap R_5$. |
| e) $R_1 - R_2$. | f) $R_2 - R_1$. |
| g) $R_1 \oplus R_3$. | h) $R_2 \oplus R_4$. |

35. Find

- | | |
|-----------------------|-----------------------|
| a) $R_2 \cup R_4$. | b) $R_3 \cup R_6$. |
| c) $R_3 \cap R_6$. | d) $R_4 \cap R_6$. |
| e) $R_3 - R_6$. | f) $R_6 - R_3$. |
| g) $R_2 \oplus R_6$. | h) $R_3 \oplus R_5$. |

36. Find

- | | |
|----------------------|----------------------|
| a) $R_1 \circ R_1$. | b) $R_1 \circ R_2$. |
| c) $R_1 \circ R_3$. | d) $R_1 \circ R_4$. |
| e) $R_1 \circ R_5$. | f) $R_1 \circ R_6$. |
| g) $R_2 \circ R_3$. | h) $R_3 \circ R_3$. |

37. Find

- | | |
|----------------------|----------------------|
| a) $R_2 \circ R_1$. | b) $R_2 \circ R_2$. |
| c) $R_3 \circ R_5$. | d) $R_4 \circ R_1$. |
| e) $R_5 \circ R_3$. | f) $R_3 \circ R_6$. |
| g) $R_4 \circ R_6$. | h) $R_6 \circ R_6$. |

38. Let R be the parent relation on the set of all people (see Example 21). When is an ordered pair in the relation R^3 ?

39. Let R be the relation on the set of people with doctorates such that $(a, b) \in R$ if and only if a was the thesis advisor of b . When is an ordered pair (a, b) in R^2 ? When is an ordered pair (a, b) in R^n , when n is a positive integer? (Assume that every person with a doctorate has a thesis advisor.)

40. Let R_1 and R_2 be the “divides” and “is a multiple of” relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$. Find

- | | |
|-----------------------|---------------------|
| a) $R_1 \cup R_2$. | b) $R_1 \cap R_2$. |
| c) $R_1 - R_2$. | d) $R_2 - R_1$. |
| e) $R_1 \oplus R_2$. | |

41. Let R_1 and R_2 be the “congruent modulo 3” and the “congruent modulo 4” relations, respectively, on the set of integers. That is, $R_1 = \{(a, b) \mid a \equiv b \pmod{3}\}$ and $R_2 = \{(a, b) \mid a \equiv b \pmod{4}\}$. Find
- $R_1 \cup R_2$.
 - $R_1 \cap R_2$.
 - $R_1 - R_2$.
 - $R_2 - R_1$.
 - $R_1 \oplus R_2$.
42. List the 16 different relations on the set $\{0, 1\}$.
43. How many of the 16 different relations on $\{0, 1\}$ contain the pair $(0, 1)$?
44. Which of the 16 relations on $\{0, 1\}$, which you listed in Exercise 42, are
- reflexive?
 - irreflexive?
 - symmetric?
 - antisymmetric?
 - asymmetric?
 - transitive?
45. a) How many relations are there on the set $\{a, b, c, d\}$?
 b) How many relations are there on the set $\{a, b, c, d\}$ that contain the pair (a, a) ?
46. Let S be a set with n elements and let a and b be distinct elements of S . How many relations R are there on S such that
- $(a, b) \in R$?
 - $(a, b) \notin R$?
 - no ordered pair in R has a as its first element?
 - at least one ordered pair in R has a as its first element?
 - no ordered pair in R has a as its first element or b as its second element?
 - at least one ordered pair in R either has a as its first element or has b as its second element?
- *47. How many relations are there on a set with n elements that are
- symmetric?
 - antisymmetric?
 - asymmetric?
 - irreflexive?
 - reflexive and symmetric?
 - neither reflexive nor irreflexive?
- *48. How many transitive relations are there on a set with n elements if
- $n = 1$?
 - $n = 2$?
 - $n = 3$?
49. Find the error in the “proof” of the following “theorem.”
- “*Theorem*”: Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.
- “*Proof*”: Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.
50. Suppose that R and S are reflexive relations on a set A . Prove or disprove each of these statements.
- $R \cup S$ is reflexive.
 - $R \cap S$ is reflexive.
 - $R \oplus S$ is irreflexive.
 - $R - S$ is irreflexive.
 - $S \circ R$ is reflexive.
51. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$, where R^{-1} is the inverse relation.
52. Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.
53. Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.
54. Show that the relation R on a set A is reflexive if and only if the complementary relation \bar{R} is irreflexive.
55. Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n .
56. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2),$ and $(5, 4)$. Find
- R^2 .
 - R^3 .
 - R^4 .
 - R^5 .
57. Let R be a reflexive relation on a set A . Show that R^n is reflexive for all positive integers n .
- *58. Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n .
59. Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

9.2 n -ary Relations and Their Applications

Introduction

Relationships among elements of more than two sets often arise. For instance, there is a relationship involving the name of a student, the student’s major, and the student’s grade point average. Similarly, there is a relationship involving the airline, flight number, starting point, destination, departure time, and arrival time of a flight. An example of such a relationship in mathematics involves three integers, where the first integer is larger than the second integer, which is larger than the third. Another example is the betweenness relationship involving points on a line, such that three points are related when the second point is between the first and the third.

We will study relationships among elements from more than two sets in this section. These relationships are called **n -ary relations**. These relations are used to represent computer databases. These representations help us answer queries about the information stored in databases, such as: Which flights land at O’Hare Airport between 3 A.M. and 4 A.M.? Which students at your

TABLE 8 Flights.				
<i>Airline</i>	<i>Flight_number</i>	<i>Gate</i>	<i>Destination</i>	<i>Departure_time</i>
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

and 09:44. SQL uses the FROM clause to identify the n -ary relation the query is applied to, the WHERE clause to specify the condition of the selection operation, and the SELECT clause to specify the projection operation that is to be applied. (*Beware:* SQL uses SELECT to represent a projection, rather than a selection operation. This is an unfortunate example of conflicting terminology.)

Example 13 shows how SQL queries can be made involving more than one table.

EXAMPLE 13 The SQL statement

```
SELECT Professor, Time
FROM Teaching_assignments, Class_schedule
WHERE Department='Mathematics'
```

is used to find the projection $P_{1,5}$ of the 5-tuples in the database (shown in Table 7), which is the join J_2 of the Teaching_assignments and Class_schedule databases in Tables 5 and 6, respectively, which satisfy the condition: Department = Mathematics. The output would consist of the single 2-tuple (Rosen, 3:00 P.M.). The SQL FROM clause is used here to find the join of two different databases.

We have only touched on the basic concepts of relational databases in this section. More information can be found in [AhUI95].

Exercises

- List the triples in the relation $\{(a, b, c) \mid a, b, \text{ and } c \text{ are integers with } 0 < a < b < c < 5\}$.
- Which 4-tuples are in the relation $\{(a, b, c, d) \mid a, b, c, \text{ and } d \text{ are positive integers with } abcd = 6\}$?
- List the 5-tuples in the relation in Table 8.
- Assuming that no new n -tuples are added, find all the primary keys for the relations displayed in
 - Table 3.
 - Table 5.
 - Table 6.
 - Table 8.
- Assuming that no new n -tuples are added, find a composite key with two fields containing the *Airline* field for the database in Table 8.
- Assuming that no new n -tuples are added, find a composite key with two fields containing the *Professor* field for the database in Table 7.
- The 3-tuples in a 3-ary relation represent the following attributes of a student database: student ID number, name, phone number.
 - Is student ID number likely to be a primary key?
 - Is name likely to be a primary key?
 - Is phone number likely to be a primary key?
- The 4-tuples in a 4-ary relation represent these attributes of published books: title, ISBN, publication date, number of pages.
 - What is a likely primary key for this relation?
 - Under what conditions would (title, publication date) be a composite key?
 - Under what conditions would (title, number of pages) be a composite key?

9. The 5-tuples in a 5-ary relation represent these attributes of all people in the United States: name, Social Security number, street address, city, state.
- Determine a primary key for this relation.
 - Under what conditions would (name, street address) be a composite key?
 - Under what conditions would (name, street address, city) be a composite key?
10. What do you obtain when you apply the selection operator s_C , where C is the condition Room = A100, to the database in Table 7?
11. What do you obtain when you apply the selection operator s_C , where C is the condition Destination = Detroit, to the database in Table 8?
12. What do you obtain when you apply the selection operator s_C , where C is the condition (Project = 2) \wedge (Quantity \geq 50), to the database in Table 10?
13. What do you obtain when you apply the selection operator s_C , where C is the condition (Airline = Nadir) \vee (Destination = Denver), to the database in Table 8?
14. What do you obtain when you apply the projection $P_{2,3,5}$ to the 5-tuple (a, b, c, d, e) ?
15. Which projection mapping is used to delete the first, second, and fourth components of a 6-tuple?
16. Display the table produced by applying the projection $P_{1,2,4}$ to Table 8.
17. Display the table produced by applying the projection $P_{1,4}$ to Table 8.
18. How many components are there in the n -tuples in the table obtained by applying the join operator J_3 to two tables with 5-tuples and 8-tuples, respectively?
19. Construct the table obtained by applying the join operator J_2 to the relations in Tables 9 and 10.
20. Show that if C_1 and C_2 are conditions that elements of the n -ary relation R may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.
21. Show that if C_1 and C_2 are conditions that elements of the n -ary relation R may satisfy, then $s_{C_1}(s_{C_2}(R)) = s_{C_2}(s_{C_1}(R))$.
22. Show that if C is a condition that elements of the n -ary relations R and S may satisfy, then $s_C(R \cup S) = s_C(R) \cup s_C(S)$.
23. Show that if C is a condition that elements of the n -ary relations R and S may satisfy, then $s_C(R \cap S) = s_C(R) \cap s_C(S)$.
24. Show that if C is a condition that elements of the n -ary relations R and S may satisfy, then $s_C(R - S) = s_C(R) - s_C(S)$.
25. Show that if R and S are both n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R \cup S) = P_{i_1, i_2, \dots, i_m}(R) \cup P_{i_1, i_2, \dots, i_m}(S)$.
26. Give an example to show that if R and S are both n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R \cap S)$ may be different from $P_{i_1, i_2, \dots, i_m}(R) \cap P_{i_1, i_2, \dots, i_m}(S)$.
27. Give an example to show that if R and S are both n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R - S)$ may be different from $P_{i_1, i_2, \dots, i_m}(R) - P_{i_1, i_2, \dots, i_m}(S)$.
28. a) What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier
FROM Part_needs
WHERE 1000 ≤ Part_number ≤ 5000
```

- b) What is the output of this query given the database in Table 9 as input?

29. a) What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier, Project
FROM Part_needs, Parts_inventory
WHERE Quantity ≤ 10
```

- b) What is the output of this query given the databases in Tables 9 and 10 as input?

30. Determine whether there is a primary key for the relation in Example 2.
31. Determine whether there is a primary key for the relation in Example 3.
32. Show that an n -ary relation with a primary key can be thought of as the graph of a function that maps values of the primary key to $(n - 1)$ -tuples formed from values of the other domains.

TABLE 9 Part_needs.		
Supplier	Part_number	Project
23	1092	1
23	1101	3
23	9048	4
31	4975	3
31	3477	2
32	6984	4
32	9191	2
33	1001	1

TABLE 10 Parts_inventory.			
Part_number	Project	Quantity	Color_code
1001	1	14	8
1092	1	2	2
1101	3	1	1
3477	2	25	2
4975	3	6	2
6984	4	10	1
9048	4	12	2
9191	2	80	4

9.3 Representing Relations

Introduction

In this section, and in the remainder of this chapter, all relations we study will be binary relations. Because of this, in this section and in the rest of this chapter, the word relation will always refer to a binary relation. There are many ways to represent a relation between finite sets. As we have seen in Section 9.1, one way is to list its ordered pairs. Another way to represent a relation is to use a table, as we did in Example 3 in Section 9.1. In this section we will discuss two alternative methods for representing relations. One method uses zero–one matrices. The other method uses pictorial representations called directed graphs, which we will discuss later in this section.

Generally, matrices are appropriate for the representation of relations in computer programs. On the other hand, people often find the representation of relations using directed graphs useful for understanding the properties of these relations.

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B .) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$


In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j , and a 0 in this position if a_i is not related to b_j . (Such a representation depends on the orderings used for A and B .)

The use of matrices to represent relations is illustrated in Examples 1–6.

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?


Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in \mathbf{M}_R show that the pairs $(2, 1)$, $(3, 1)$, and $(3, 2)$ belong to R . The 0s show that no other pairs belong to R . 

EXAMPLE 2 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Because loops are not present at all the vertices of the directed graph of S , this relation is not reflexive. It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S is not transitive, because (c, a) and (a, b) belong to S , but (c, b) does not belong to S . 

Exercises

- Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).
 - $\{(1, 1), (1, 2), (1, 3)\}$
 - $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
 - $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
 - $\{(1, 3), (3, 1)\}$
- Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).
 - $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
 - $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
 - $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
 - $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$
- List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).
 - $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
 - $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
 - $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).
 - $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$
 - $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$
 - $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$
- How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?
- How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?
- Determine whether the relations represented by the matrices in Exercise 3 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.
- Determine whether the relations represented by the matrices in Exercise 4 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

- How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \dots, 100\}$ consisting of the first 100 positive integers have if R is
 - $\{(a, b) \mid a > b\}$
 - $\{(a, b) \mid a \neq b\}$
 - $\{(a, b) \mid a = b + 1\}$
 - $\{(a, b) \mid a = 1\}$
 - $\{(a, b) \mid ab = 1\}$
- How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \dots, 1000\}$ consisting of the first 1000 positive integers have if R is
 - $\{(a, b) \mid a \leq b\}$
 - $\{(a, b) \mid a = b \pm 1\}$
 - $\{(a, b) \mid a + b = 1000\}$
 - $\{(a, b) \mid a + b \leq 1001\}$
 - $\{(a, b) \mid a \neq 0\}$
- How can the matrix for \bar{R} , the complement of the relation R , be found from the matrix representing R , when R is a relation on a finite set A ?
- How can the matrix for R^{-1} , the inverse of the relation R , be found from the matrix representing R , when R is a relation on a finite set A ?
- Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

- R^{-1} .
 - \bar{R} .
 - R^2 .
14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

- $R_1 \cup R_2$.
 - $R_1 \cap R_2$.
 - $R_2 \circ R_1$.
 - $R_1 \circ R_1$.
 - $R_1 \oplus R_2$.
15. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

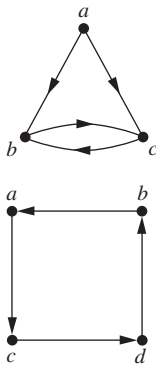
Find the matrices that represent

- R^2 .
- R^3 .
- R^4 .

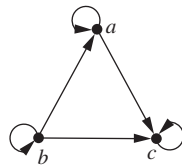
16. Let R be a relation on a set A with n elements. If there are k nonzero entries in \mathbf{M}_R , the matrix representing R , how many nonzero entries are there in $\mathbf{M}_{R^{-1}}$, the matrix representing R^{-1} , the inverse of R ?
17. Let R be a relation on a set A with n elements. If there are k nonzero entries in \mathbf{M}_R , the matrix representing R , how many nonzero entries are there in $\mathbf{M}_{\bar{R}}$, the matrix representing \bar{R} , the complement of R ?
18. Draw the directed graphs representing each of the relations from Exercise 1.
19. Draw the directed graphs representing each of the relations from Exercise 2.
20. Draw the directed graph representing each of the relations from Exercise 3.
21. Draw the directed graph representing each of the relations from Exercise 4.
22. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

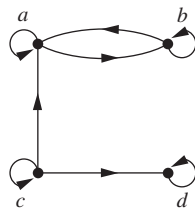
23.



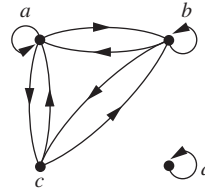
24.



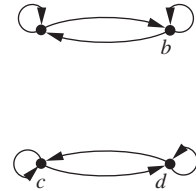
25.



27.



28.



29. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is asymmetric?
30. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is irreflexive?
31. Determine whether the relations represented by the directed graphs shown in Exercises 23–25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.
32. Determine whether the relations represented by the directed graphs shown in Exercises 26–28 are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.
33. Let R be a relation on a set A . Explain how to use the directed graph representing R to obtain the directed graph representing the inverse relation R^{-1} .
34. Let R be a relation on a set A . Explain how to use the directed graph representing R to obtain the directed graph representing the complementary relation \bar{R} .
35. Show that if \mathbf{M}_R is the matrix representing the relation R , then $\mathbf{M}_R^{[n]}$ is the matrix representing the relation R^n .
36. Given the directed graphs representing two relations, how can the directed graph of the union, intersection, symmetric difference, difference, and composition of these relations be found?

9.4 Closures of Relations

Introduction

A computer network has data centers in Boston, Chicago, Denver, Detroit, New York, and San Diego. There are direct, one-way telephone lines from Boston to Chicago, from Boston to Detroit, from Chicago to Detroit, from Detroit to Denver, and from New York to San Diego. Let R be the relation containing (a, b) if there is a telephone line from the data center in a to that in b . How can we determine if there is some (possibly indirect) link composed of one or more telephone lines from one center to another? Because not all links are direct, such as the link from Boston to Denver that goes through Detroit, R cannot be used directly to answer this. In the language of relations, R is not transitive, so it does not contain all the pairs that can be linked. As we will show in this section, we can find all pairs of data centers that have a link by constructing a transitive relation S containing R such that S is a subset of every transitive relation containing R . Here, S is the smallest transitive relation that contains R . This relation is called the **transitive closure** of R .

In general, let R be a relation on a set A . R may or may not have some property \mathbf{P} , such as reflexivity, symmetry, or transitivity. If there is a relation S with property \mathbf{P} containing R such that S is a subset of every relation with property \mathbf{P} containing R , then S is called the **closure**

LEMMA 2

Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j) th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \dots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j , and k are positive integers not exceeding n .

Lemma 2 gives us the means to compute efficiently the matrices $\mathbf{W}_k, k = 1, 2, \dots, n$. We display the pseudocode for Warshall's algorithm, using Lemma 2, as Algorithm 2.

ALGORITHM 2 Warshall Algorithm.

```

procedure Warshall ( $\mathbf{M}_R : n \times n$  zero–one matrix)
 $\mathbf{W} := \mathbf{M}_R$ 
for  $k := 1$  to  $n$ 
    for  $i := 1$  to  $n$ 
        for  $j := 1$  to  $n$ 
             $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$ 
return  $\mathbf{W}$  ( $\mathbf{W} = [w_{ij}]$  is  $\mathbf{M}_{R^*}$ )

```

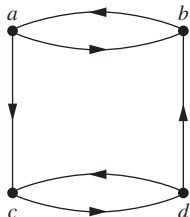
The computational complexity of Warshall's algorithm can easily be computed in terms of bit operations. To find the entry $w_{ij}^{[k]}$ from the entries $w_{ij}^{[k-1]}$, $w_{ik}^{[k-1]}$, and $w_{kj}^{[k-1]}$ using Lemma 2 requires two bit operations. To find all n^2 entries of \mathbf{W}_k from those of \mathbf{W}_{k-1} requires $2n^2$ bit operations. Because Warshall's algorithm begins with $\mathbf{W}_0 = \mathbf{M}_R$ and computes the sequence of n zero–one matrices $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_n = \mathbf{M}_{R^*}$, the total number of bit operations used is $n \cdot 2n^2 = 2n^3$.

Exercises

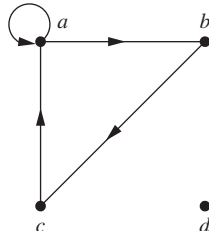
- Let R be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 2)$, and $(3, 0)$. Find the
 - reflexive closure of R .
 - symmetric closure of R .
- Let R be the relation $\{(a, b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R ?
- Let R be the relation $\{(a, b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R ?
- How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

In Exercises 5–7 draw the directed graph of the reflexive closure of the relations with the directed graph shown.

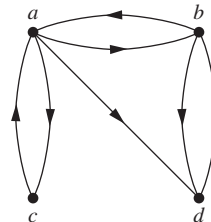
5.



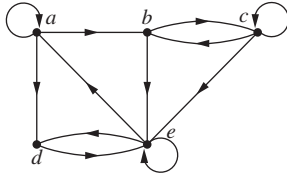
6.



7.



- How can the directed graph representing the symmetric closure of a relation on a finite set be constructed from the directed graph for this relation?
- Find the directed graphs of the symmetric closures of the relations with directed graphs shown in Exercises 5–7.
- Find the smallest relation containing the relation in Example 2 that is both reflexive and symmetric.
- Find the directed graph of the smallest relation that is both reflexive and symmetric that contains each of the relations with directed graphs shown in Exercises 5–7.
- Suppose that the relation R on the finite set A is represented by the matrix \mathbf{M}_R . Show that the matrix that represents the reflexive closure of R is $\mathbf{M}_R \vee \mathbf{I}_n$.

13. Suppose that the relation R on the finite set A is represented by the matrix \mathbf{M}_R . Show that the matrix that represents the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}'_R$.
14. Show that the closure of a relation R with respect to a property \mathbf{P} , if it exists, is the intersection of all the relations with property \mathbf{P} that contain R .
15. When is it possible to define the “irreflexive closure” of a relation R , that is, a relation that contains R , is irreflexive, and is contained in every irreflexive relation that contains R ?
16. Determine whether these sequences of vertices are paths in this directed graph.
- 
- a) a, b, c, e
 b) b, e, c, b, e
 c) a, a, b, e, d, e
 d) b, c, e, d, a, a, b
 e) b, c, c, b, e, d, e, d
 f) $a, a, b, b, c, c, b, e, d$
17. Find all circuits of length three in the directed graph in Exercise 16.
18. Determine whether there is a path in the directed graph in Exercise 16 beginning at the first vertex given and ending at the second vertex given.
- | | | |
|-----------|-----------|-----------|
| a) a, b | b) b, a | c) b, b |
| d) a, e | e) b, d | f) c, d |
| g) d, d | h) e, a | i) e, c |
19. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), (5, 2)$, and $(5, 4)$. Find
- | | | |
|------------|------------|------------|
| a) R^2 . | b) R^3 . | c) R^4 . |
| d) R^5 . | e) R^6 . | f) R^* . |
20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b . When is (a, b) in
- | | | |
|------------|------------|------------|
| a) R^2 ? | b) R^3 ? | c) R^* ? |
|------------|------------|------------|
21. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in
- | | | |
|------------|------------|------------|
| a) R^2 ? | b) R^3 ? | c) R^* ? |
|------------|------------|------------|
22. Suppose that the relation R is reflexive. Show that R^* is reflexive.
23. Suppose that the relation R is symmetric. Show that R^* is symmetric.
24. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?
25. Use Algorithm 1 to find the transitive closures of these relations on $\{1, 2, 3, 4\}$.
- | |
|---|
| a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$ |
| b) $\{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$ |
| c) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$ |
| d) $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$ |
26. Use Algorithm 1 to find the transitive closures of these relations on $\{a, b, c, d, e\}$.
- | |
|---|
| a) $\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$ |
| b) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$ |
| c) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$ |
| d) $\{(a, e), (b, a), (b, d), (c, d), (d, a), (d, c), (e, a), (e, b), (e, c), (e, e)\}$ |
27. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 25.
28. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 26.
29. Find the smallest relation containing the relation $\{(1, 2), (1, 4), (3, 3), (4, 1)\}$ that is
- | |
|--|
| a) reflexive and transitive. |
| b) symmetric and transitive. |
| c) reflexive, symmetric, and transitive. |
30. Finish the proof of the case when $a \neq b$ in Lemma 1.
31. Algorithms have been devised that use $O(n^{2.8})$ bit operations to compute the Boolean product of two $n \times n$ zero-one matrices. Assuming that these algorithms can be used, give big- O estimates for the number of bit operations using Algorithm 1 and using Warshall's algorithm to find the transitive closure of a relation on a set with n elements.
- *32. Devise an algorithm using the concept of interior vertices in a path to find the length of the shortest path between two vertices in a directed graph, if such a path exists.
33. Adapt Algorithm 1 to find the reflexive closure of the transitive closure of a relation on a set with n elements.
34. Adapt Warshall's algorithm to find the reflexive closure of the transitive closure of a relation on a set with n elements.
35. Show that the closure with respect to the property \mathbf{P} of the relation $R = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$ on the set $\{0, 1, 2\}$ does not exist if \mathbf{P} is the property
- | |
|-------------------------------------|
| a) “is not reflexive.” |
| b) “has an odd number of elements.” |

9.5 Equivalence Relations

Introduction

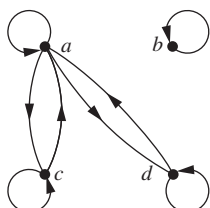
In some programming languages the names of variables can contain an unlimited number of characters. However, there is a limit on the number of characters that are checked when a compiler determines whether two variables are equal. For instance, in traditional C, only the first eight characters of a variable name are checked by the compiler. (These characters are

Exercises

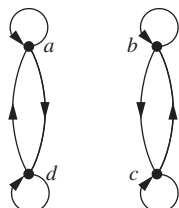
- Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.
 - $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
 - $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
 - $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
 - $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
 - $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$
- Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.
 - $\{(a, b) \mid a \text{ and } b \text{ are the same age}\}$
 - $\{(a, b) \mid a \text{ and } b \text{ have the same parents}\}$
 - $\{(a, b) \mid a \text{ and } b \text{ share a common parent}\}$
 - $\{(a, b) \mid a \text{ and } b \text{ have met}\}$
 - $\{(a, b) \mid a \text{ and } b \text{ speak a common language}\}$
- Which of these relations on the set of all functions from \mathbf{Z} to \mathbf{Z} are equivalence relations? Determine the properties of an equivalence relation that the others lack.
 - $\{(f, g) \mid f(1) = g(1)\}$
 - $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
 - $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbf{Z}\}$
 - $\{(f, g) \mid \text{for some } C \in \mathbf{Z}, \text{ for all } x \in \mathbf{Z}, f(x) - g(x) = C\}$
 - $\{(f, g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$
- Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.
- Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.
- Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.
- Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of \mathbf{F} and of \mathbf{T} ?
- Let R be the relation on the set of all sets of real numbers such that $S R T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and \mathbf{Z} ?
- Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.
 - Show that R is an equivalence relation on A .
 - What are the equivalence classes of R ?
- Suppose that A is a nonempty set and R is an equivalence relation on A . Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.
- Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.
- Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.
- Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.
- Let R be the relation consisting of all pairs (x, y) such that x and y are strings of uppercase and lowercase English letters with the property that for every positive integer n , the n th characters in x and y are the same letter, either uppercase or lowercase. Show that R is an equivalence relation.
- Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.
- Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.
- (Requires calculus)
 - Show that the relation R on the set of all differentiable functions from \mathbf{R} to \mathbf{R} consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x is an equivalence relation.
 - Which functions are in the same equivalence class as the function $f(x) = x^2$?
- (Requires calculus)
 - Let n be a positive integer. Show that the relation R on the set of all polynomials with real-valued coefficients consisting of all pairs (f, g) such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the n th derivative of $f(x)$.]
 - Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?
- Let R be the relation on the set of all URLs (or Web addresses) such that $x R y$ if and only if the Web page at x is the same as the Web page at y . Show that R is an equivalence relation.
- Let R be the relation on the set of all people who have visited a particular Web page such that $x R y$ if and only if person x and person y have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that R is an equivalence relation.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

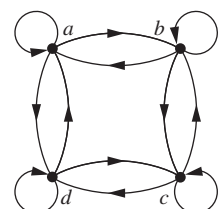
21.



22.



23.



24. Determine whether the relations represented by these zero-one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. Show that the relation R on the set of all bit strings such that $s R t$ if and only if s and t contain the same number of 1s is an equivalence relation.

26. What are the equivalence classes of the equivalence relations in Exercise 1?

27. What are the equivalence classes of the equivalence relations in Exercise 2?

28. What are the equivalence classes of the equivalence relations in Exercise 3?

29. What is the equivalence class of the bit string 011 for the equivalence relation in Exercise 25?

30. What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

a) 010 b) 1011 c) 11111 d) 01010101

31. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 12?

32. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 13?

33. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_4 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_4 if and only if they are equal or they are both at least four bits long and agree in their first four bits.)

34. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_5 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_5 if and only if they are equal or they are both at least five bits long and agree in their first five bits.)

35. What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2? b) 3? c) 6? d) -3 ?

36. What is the congruence class $[4]_m$ when m is

a) 2? b) 3? c) 6? d) 8?

37. Give a description of each of the congruence classes modulo 6.

38. What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

a) No b) Yes c) Help

39. a) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 15?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 15. [Hint: Look at the difference $a - b$ corresponding to (a, b) .]

40. a) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 16?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 16. [Hint: Look at the ratio a/b corresponding to (a, b) .]

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$ b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$ d) $\{1, 4, 5\}, \{2, 6\}$

42. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

43. Which of these collections of subsets are partitions of the set of bit strings of length 8?

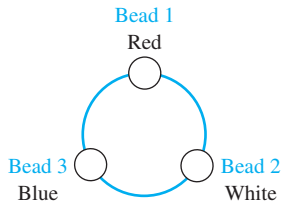
a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01
b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11
c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11
d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00
e) the set of bit strings that contain $3k$ ones for some nonnegative integer k ; the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k ; and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k .

44. Which of these collections of subsets are partitions of the set of integers?

a) the set of even integers and the set of odd integers
b) the set of positive integers and the set of negative integers

- c) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3
- d) the set of integers less than -100 , the set of integers with absolute value not exceeding 100, and the set of integers greater than 100
- e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6
45. Which of these are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?
- a) the set of pairs (x, y) , where x or y is odd; the set of pairs (x, y) , where x is even; and the set of pairs (x, y) , where y is even
- b) the set of pairs (x, y) , where both x and y are odd; the set of pairs (x, y) , where exactly one of x and y is odd; and the set of pairs (x, y) , where both x and y are even
- c) the set of pairs (x, y) , where x is positive; the set of pairs (x, y) , where y is positive; and the set of pairs (x, y) , where both x and y are negative
- d) the set of pairs (x, y) , where $3 \mid x$ and $3 \mid y$; the set of pairs (x, y) , where $3 \mid x$ and $3 \nmid y$; the set of pairs (x, y) , where $3 \nmid x$ and $3 \mid y$; and the set of pairs (x, y) , where $3 \nmid x$ and $3 \nmid y$
- e) the set of pairs (x, y) , where $x > 0$ and $y > 0$; the set of pairs (x, y) , where $x > 0$ and $y \leq 0$; the set of pairs (x, y) , where $x \leq 0$ and $y > 0$; and the set of pairs (x, y) , where $x \leq 0$ and $y \leq 0$
- f) the set of pairs (x, y) , where $x \neq 0$ and $y \neq 0$; the set of pairs (x, y) , where $x = 0$ and $y \neq 0$; and the set of pairs (x, y) , where $x \neq 0$ and $y = 0$
46. Which of these are partitions of the set of real numbers?
- a) the negative real numbers, $\{0\}$, the positive real numbers
- b) the set of irrational numbers, the set of rational numbers
- c) the set of intervals $[k, k + 1]$, $k = \dots, -2, -1, 0, 1, 2, \dots$
- d) the set of intervals $(k, k + 1)$, $k = \dots, -2, -1, 0, 1, 2, \dots$
- e) the set of intervals $(k, k + 1]$, $k = \dots, -2, -1, 0, 1, 2, \dots$
- f) the sets $\{x + n \mid n \in \mathbf{Z}\}$ for all $x \in [0, 1)$
47. List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.
- a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$
- b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$
- c) $\{0, 1, 2\}, \{3, 4, 5\}$
- d) $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$
48. List the ordered pairs in the equivalence relations produced by these partitions of $\{a, b, c, d, e, f, g\}$.
- a) $\{a, b\}, \{c, d\}, \{e, f, g\}$
- b) $\{a\}, \{b\}, \{c, d\}, \{e, f\}, \{g\}$
- c) $\{a, b, c, d\}, \{e, f, g\}$
- d) $\{a, c, e, g\}, \{b, d\}, \{f\}$
- A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2 .
49. Show that the partition formed from congruence classes modulo 6 is a refinement of the partition formed from congruence classes modulo 3.
50. Show that the partition of the set of people living in the United States consisting of subsets of people living in the same county (or parish) and same state is a refinement of the partition consisting of subsets of people living in the same state.
51. Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last four bits.
- In Exercises 52 and 53, R_n refers to the family of equivalence relations defined in Example 5. Recall that $s R_n t$, where s and t are two strings if $s = t$ or s and t are strings with at least n characters that agree in their first n characters.
52. Show that the partition of the set of all bit strings formed by equivalence classes of bit strings with respect to the equivalence relation R_4 is a refinement of the partition formed by equivalence classes of bit strings with respect to the equivalence relation R_3 .
53. Show that the partition of the set of all identifiers in C formed by the equivalence classes of identifiers with respect to the equivalence relation R_{31} is a refinement of the partition formed by equivalence classes of identifiers with respect to the equivalence relation R_8 . (Compilers for “old” C consider identifiers the same when their names agree in their first eight characters, while compilers in standard C consider identifiers the same when their names agree in their first 31 characters.)
54. Suppose that R_1 and R_2 are equivalence relations on a set A . Let P_1 and P_2 be the partitions that correspond to R_1 and R_2 , respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2 .
55. Find the smallest equivalence relation on the set $\{a, b, c, d, e\}$ containing the relation $\{(a, b), (a, c), (d, e)\}$.
56. Suppose that R_1 and R_2 are equivalence relations on the set S . Determine whether each of these combinations of R_1 and R_2 must be an equivalence relation.
- a) $R_1 \cup R_2$ b) $R_1 \cap R_2$ c) $R_1 \oplus R_2$
57. Consider the equivalence relation from Example 2, namely, $R = \{(x, y) \mid x - y \text{ is an integer}\}$.
- a) What is the equivalence class of 1 for this equivalence relation?
- b) What is the equivalence class of $1/2$ for this equivalence relation?

- *58. Each bead on a bracelet with three beads is either red, white, or blue, as illustrated in the figure shown.



Define the relation R between bracelets as: (B_1, B_2) , where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

- a) Show that R is an equivalence relation.
 - b) What are the equivalence classes of R ?
- *59. Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C_1, C_2) , where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.
- a) Show that R is an equivalence relation.
 - b) What are the equivalence classes of R ?
60. a) Let R be the relation on the set of functions from \mathbf{Z}^+ to \mathbf{Z}^+ such that (f, g) belongs to R if and only if f is $\Theta(g)$ (see Section 3.2). Show that R is an equivalence relation.
- b) Describe the equivalence class containing $f(n) = n^2$ for the equivalence relation of part (a).
61. Determine the number of different equivalence relations on a set with three elements by listing them.
62. Determine the number of different equivalence relations on a set with four elements by listing them.
- *63. Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?
- *64. Do we necessarily get an equivalence relation when we form the symmetric closure of the reflexive closure of the transitive closure of a relation?
65. Suppose we use Theorem 2 to form a partition P from an equivalence relation R . What is the equivalence relation R' that results if we use Theorem 2 again to form an equivalence relation from P ?
66. Suppose we use Theorem 2 to form an equivalence relation R from a partition P . What is the partition P' that results if we use Theorem 2 again to form a partition from R ?
67. Devise an algorithm to find the smallest equivalence relation containing a given relation.
- *68. Let $p(n)$ denote the number of different equivalence relations on a set with n elements (and by Theorem 2 the number of partitions of a set with n elements). Show that $p(n)$ satisfies the recurrence relation $p(n) = \sum_{j=0}^{n-1} C(n-1, j)p(n-j-1)$ and the initial condition $p(0) = 1$. (Note: The numbers $p(n)$ are called **Bell numbers** after the American mathematician E. T. Bell.)
69. Use Exercise 68 to find the number of different equivalence relations on a set with n elements, where n is a positive integer not exceeding 10.

9.6 Partial Orderings

Introduction

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y) , where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y) , where x and y are tasks in a project such that x must be completed before y begins. We order the set of integers using the relation containing the pairs (x, y) , where x is less than y . When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.



DEFINITION 1

A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R) . Members of S are called *elements* of the poset.

We give examples of posets in Examples 1–3.

EXAMPLE 1 Show that the “greater than or equal” relation (\geq) is a partial ordering on the set of integers.

Exercises

- Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.
 - $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
 - $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
 - $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$
 - $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
 - $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$
- Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.
 - $\{(0, 0), (2, 2), (3, 3)\}$
 - $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$
 - $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$
 - $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$
 - $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$
- Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if
 - a is taller than b ?
 - a is not taller than b ?
 - $a = b$ or a is an ancestor of b ?
 - a and b have a common friend?
- Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if
 - a is no shorter than b ?
 - a weighs more than b ?
 - $a = b$ or a is a descendant of b ?
 - a and b do not have a common friend?
- Which of these are posets?
 - $(\mathbf{Z}, =)$
 - (\mathbf{Z}, \neq)
 - (\mathbf{Z}, \geq)
 - (\mathbf{Z}, \nmid)
- Which of these are posets?
 - $(\mathbf{R}, =)$
 - $(\mathbf{R}, <)$
 - (\mathbf{R}, \leq)
 - (\mathbf{R}, \neq)
- Determine whether the relations represented by these zero-one matrices are partial orders.

$$\text{a) } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b) } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{c) } \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

- Determine whether the relations represented by these zero-one matrices are partial orders.

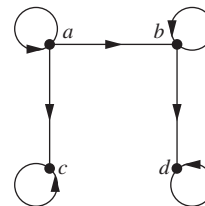
$$\text{a) } \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b) } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

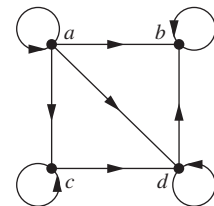
$$\text{c) } \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

In Exercises 9–11 determine whether the relation with the directed graph shown is a partial order.

9.



10.



11.

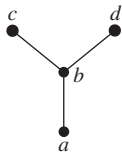


- Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse of R . The poset (S, R^{-1}) is called the **dual** of (S, R) .
- Find the duals of these posets.
 - $(\{0, 1, 2\}, \leq)$
 - (\mathbf{Z}, \geq)
 - $(P(\mathbf{Z}), \supseteq)$
 - $(\mathbf{Z}^+, |)$
- Which of these pairs of elements are comparable in the poset $(\mathbf{Z}^+, |)$?
 - 5, 15
 - 6, 9
 - 8, 16
 - 7, 7
- Find two incomparable elements in these posets.
 - $(P(\{0, 1, 2\}), \subseteq)$
 - $(\{1, 2, 4, 6, 8\}, |)$
- Let $S = \{1, 2, 3, 4\}$. With respect to the lexicographic order based on the usual “less than” relation,
 - find all pairs in $S \times S$ less than $(2, 3)$.
 - find all pairs in $S \times S$ greater than $(3, 1)$.
 - draw the Hasse diagram of the poset $(S \times S, \preceq)$.
- Find the lexicographic ordering of these n -tuples:
 - $(1, 1, 2), (1, 2, 1)$
 - $(0, 1, 2, 3), (0, 1, 3, 2)$
 - $(1, 0, 1, 0, 1), (0, 1, 1, 1, 0)$
- Find the lexicographic ordering of these strings of lowercase English letters:
 - quack, quick, quicksilver, quicksand, quacking*
 - open, opener, opera, operand, opened*
 - zoo, zero, zoom, zoology, zoological*
- Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 < 1$.
- Draw the Hasse diagram for the “greater than or equal to” relation on $\{0, 1, 2, 3, 4, 5\}$.

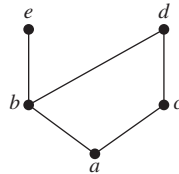
21. Draw the Hasse diagram for the “less than or equal to” relation on $\{0, 2, 5, 10, 11, 15\}$.
22. Draw the Hasse diagram for divisibility on the set
 a) $\{1, 2, 3, 4, 5, 6\}$. b) $\{3, 5, 7, 11, 13, 16, 17\}$.
 c) $\{2, 3, 5, 10, 11, 15, 25\}$. d) $\{1, 3, 9, 27, 81, 243\}$.
23. Draw the Hasse diagram for divisibility on the set
 a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$. b) $\{1, 2, 3, 5, 7, 11, 13\}$.
 c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$.
 d) $\{1, 2, 4, 8, 16, 32, 64\}$.
24. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$.

In Exercises 25–27 list all ordered pairs in the partial ordering with the accompanying Hasse diagram.

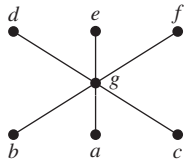
25.



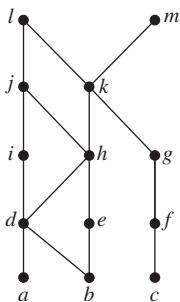
26.



27.



28. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 12\}$?
29. What is the covering relation of the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set of S , where $S = \{a, b, c\}$?
30. What is the covering relation of the partial ordering for the poset of security classes defined in Example 25?
31. Show that a finite poset can be reconstructed from its covering relation. [Hint: Show that the poset is the reflexive transitive closure of its covering relation.]
32. Answer these questions for the partial order represented by this Hasse diagram.



- a) Find the maximal elements.
 b) Find the minimal elements.
 c) Is there a greatest element?

- d) Is there a least element?
 e) Find all upper bounds of $\{a, b, c\}$.
 f) Find the least upper bound of $\{a, b, c\}$, if it exists.
 g) Find all lower bounds of $\{f, g, h\}$.
 h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.
33. Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, \mid)$.
 a) Find the maximal elements.
 b) Find the minimal elements.
 c) Is there a greatest element?
 d) Is there a least element?
 e) Find all upper bounds of $\{3, 5\}$.
 f) Find the least upper bound of $\{3, 5\}$, if it exists.
 g) Find all lower bounds of $\{15, 45\}$.
 h) Find the greatest lower bound of $\{15, 45\}$, if it exists.
34. Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, \mid)$.
 a) Find the maximal elements.
 b) Find the minimal elements.
 c) Is there a greatest element?
 d) Is there a least element?
 e) Find all upper bounds of $\{2, 9\}$.
 f) Find the least upper bound of $\{2, 9\}$, if it exists.
 g) Find all lower bounds of $\{60, 72\}$.
 h) Find the greatest lower bound of $\{60, 72\}$, if it exists.
35. Answer these questions for the poset $(\{1, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$.
 a) Find the maximal elements.
 b) Find the minimal elements.
 c) Is there a greatest element?
 d) Is there a least element?
 e) Find all upper bounds of $\{\{2\}, \{4\}\}$.
 f) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.
 g) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.
 h) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, if it exists.

36. Give a poset that has

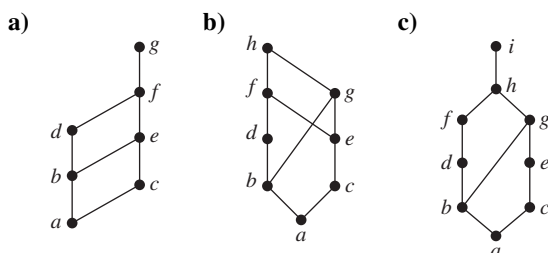
- a) a minimal element but no maximal element.
 b) a maximal element but no minimal element.
 c) neither a maximal nor a minimal element.

37. Show that lexicographic order is a partial ordering on the Cartesian product of two posets.

38. Show that lexicographic order is a partial ordering on the set of strings from a poset.

39. Suppose that (S, \preceq_1) and (T, \preceq_2) are posets. Show that $(S \times T, \preceq)$ is a poset where $(s, t) \preceq (u, v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$.

40. a) Show that there is exactly one greatest element of a poset, if such an element exists.
 b) Show that there is exactly one least element of a poset, if such an element exists.
41. a) Show that there is exactly one maximal element in a poset with a greatest element.
 b) Show that there is exactly one minimal element in a poset with a least element.
42. a) Show that the least upper bound of a set in a poset is unique if it exists.
 b) Show that the greatest lower bound of a set in a poset is unique if it exists.
43. Determine whether the posets with these Hasse diagrams are lattices.



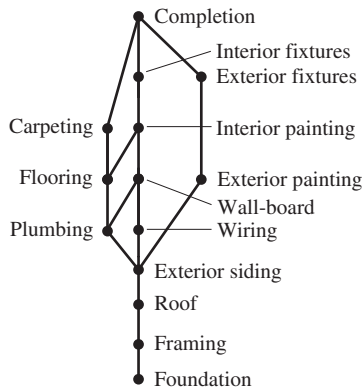
44. Determine whether these posets are lattices.
 a) $(\{1, 3, 6, 9, 12\}, |)$ b) $(\{1, 5, 25, 125\}, |)$
 c) (\mathbf{Z}, \geq)
 d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S
45. Show that every nonempty finite subset of a lattice has a least upper bound and a greatest lower bound.
46. Show that if the poset (S, R) is a lattice then the dual poset (S, R^{-1}) is also a lattice.
47. In a company, the lattice model of information flow is used to control sensitive information with security classes represented by ordered pairs (A, C) . Here A is an authority level, which may be nonproprietary (0), proprietary (1), restricted (2), or registered (3). A category C is a subset of the set of all projects $\{\text{Cheetah}, \text{Impala}, \text{Puma}\}$. (Names of animals are often used as code names for projects in companies.)
 a) Is information permitted to flow from $(\text{Proprietary}, \{\text{Cheetah}, \text{Puma}\})$ into $(\text{Restricted}, \{\text{Puma}\})$?
 b) Is information permitted to flow from $(\text{Restricted}, \{\text{Cheetah}\})$ into $(\text{Registered}, \{\text{Cheetah}, \text{Impala}\})$?
 c) Into which classes is information from $(\text{Proprietary}, \{\text{Cheetah}, \text{Puma}\})$ permitted to flow?
 d) From which classes is information permitted to flow into the security class $(\text{Restricted}, \{\text{Impala}, \text{Puma}\})$?
48. Show that the set S of security classes (A, C) is a lattice, where A is a positive integer representing an authority class and C is a subset of a finite set of compartments, with $(A_1, C_1) \preceq (A_2, C_2)$ if and only if $A_1 \leq A_2$ and $C_1 \subseteq C_2$. [Hint: First show that (S, \preceq) is a poset and then show that the least upper bound and greatest lower bound of (A_1, C_1) and (A_2, C_2) are $(\max(A_1, A_2), C_1 \cup C_2)$ and $(\min(A_1, A_2), C_1 \cap C_2)$, respectively.]

- *49. Show that the set of all partitions of a set S with the relation $P_1 \preceq P_2$ if the partition P_1 is a refinement of the partition P_2 is a lattice. (See the preamble to Exercise 49 of Section 9.5.)
50. Show that every totally ordered set is a lattice.
51. Show that every finite lattice has a least element and a greatest element.
52. Give an example of an infinite lattice with
 a) neither a least nor a greatest element.
 b) a least but not a greatest element.
 c) a greatest but not a least element.
 d) both a least and a greatest element.
53. Verify that $(\mathbf{Z}^+ \times \mathbf{Z}^+, \preceq)$ is a well-ordered set, where \preceq is lexicographic order, as claimed in Example 8.
54. Determine whether each of these posets is well-ordered.
 a) (S, \leq) , where $S = \{10, 11, 12, \dots\}$
 b) $(\mathbf{Q} \cap [0, 1], \leq)$ (the set of rational numbers between 0 and 1 inclusive)
 c) (S, \leq) , where S is the set of positive rational numbers with denominators not exceeding 3
 d) (\mathbf{Z}^-, \geq) , where \mathbf{Z}^- is the set of negative integers

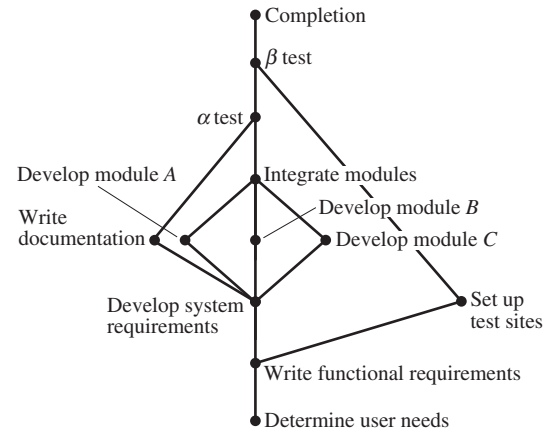
A poset (R, \preceq) is **well-founded** if there is no infinite decreasing sequence of elements in the poset, that is, elements x_1, x_2, \dots, x_n such that $\dots < x_n < \dots < x_2 < x_1$. A poset (R, \preceq) is **dense** if for all $x \in S$ and $y \in S$ with $x < y$, there is an element $z \in R$ such that $x < z < y$.

55. Show that the poset (\mathbf{Z}, \preceq) , where $x < y$ if and only if $|x| < |y|$ is well-founded but is not a totally ordered set.
56. Show that a dense poset with at least two elements that are comparable is not well-founded.
57. Show that the poset of rational numbers with the usual “less than or equal to” relation, (\mathbf{Q}, \leq) , is a dense poset.
- *58. Show that the set of strings of lowercase English letters with lexicographic order is neither well-founded nor dense.
59. Show that a poset is well-ordered if and only if it is totally ordered and well-founded.
60. Show that a finite nonempty poset has a maximal element.
61. Find a compatible total order for the poset with the Hasse diagram shown in Exercise 32.
62. Find a compatible total order for the divisibility relation on the set $\{1, 2, 3, 6, 8, 12, 24, 36\}$.
63. Find all compatible total orderings for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$ from Example 26.
64. Find all compatible total orderings for the poset with the Hasse diagram in Exercise 27.
65. Find all possible orders for completing the tasks in the development project in Example 27.

66. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing these tasks is as shown in the figure.



67. Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is as shown.



Key Terms and Results

TERMS

binary relation from A to B : a subset of $A \times B$

relation on A : a binary relation from A to itself (i.e., a subset of $A \times A$)

$S \circ R$: composite of R and S

R^{-1} : inverse relation of R

R^n : n th power of R

reflexive: a relation R on A is reflexive if $(a, a) \in R$ for all $a \in A$

symmetric: a relation R on A is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$

antisymmetric: a relation R on A is antisymmetric if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$

transitive: a relation R on A is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

n -ary relation on A_1, A_2, \dots, A_n : a subset of $A_1 \times A_2 \times \dots \times A_n$

relational data model: a model for representing databases using n -ary relations

primary key: a domain of an n -ary relation such that an n -tuple is uniquely determined by its value for this domain

composite key: the Cartesian product of domains of an n -ary relation such that an n -tuple is uniquely determined by its values in these domains

selection operator: a function that selects the n -tuples in an n -ary relation that satisfy a specified condition

projection: a function that produces relations of smaller degree from an n -ary relation by deleting fields

join: a function that combines n -ary relations that agree on certain fields

directed graph or digraph: a set of elements called vertices and ordered pairs of these elements, called edges

loop: an edge of the form (a, a)

closure of a relation R with respect to a property P : the relation S (if it exists) that contains R , has property P , and is contained within any relation that contains R and has property P

path in a digraph: a sequence of edges $(a, x_1), (x_1, x_2), \dots, (x_{n-2}, x_{n-1}), (x_{n-1}, b)$ such that the terminal vertex of each edge is the initial vertex of the succeeding edge in the sequence

circuit (or cycle) in a digraph: a path that begins and ends at the same vertex

R^* (connectivity relation): the relation consisting of those ordered pairs (a, b) such that there is a path from a to b

equivalence relation: a reflexive, symmetric, and transitive relation

equivalent: if R is an equivalence relation, a is equivalent to b if aRb

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a

$[a]_m$ (congruence class modulo m): the set of integers congruent to a modulo m

partition of a set S : a collection of pairwise disjoint nonempty subsets that have S as their union

partial ordering: a relation that is reflexive, antisymmetric, and transitive

poset (S, R) : a set S and a partial ordering R on this set

comparable: the elements a and b in the poset (A, \preceq) are comparable if $a \preceq b$ or $b \preceq a$

incomparable: elements in a poset that are not comparable

total (or linear) ordering: a partial ordering for which every pair of elements are comparable

totally (or linearly) ordered set: a poset with a total (or linear) ordering

well-ordered set: a poset (S, \preceq) , where \preceq is a total order and every nonempty subset of S has a least element

lexicographic order: a partial ordering of Cartesian products or strings

Hasse diagram: a graphical representation of a poset where loops and all edges resulting from the transitive property are not shown, and the direction of the edges is indicated by the position of the vertices

maximal element: an element of a poset that is not less than any other element of the poset

minimal element: an element of a poset that is not greater than any other element of the poset

greatest element: an element of a poset greater than all other elements in this set

least element: an element of a poset less than all other elements in this set

upper bound of a set: an element in a poset greater than all other elements in the set

lower bound of a set: an element in a poset less than all other elements in the set

least upper bound of a set: an upper bound of the set that is less than all other upper bounds

greatest lower bound of a set: a lower bound of the set that is greater than all other lower bounds

lattice: a partially ordered set in which every two elements have a greatest lower bound and a least upper bound

compatible total ordering for a partial ordering: a total ordering that contains the given partial ordering

topological sort: the construction of a total ordering compatible with a given partial ordering

RESULTS

The reflexive closure of a relation R on the set A equals $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$.

The symmetric closure of a relation R on the set A equals $R \cup R^{-1}$, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

The transitive closure of a relation equals the connectivity relation formed from this relation.

Warshall's algorithm for finding the transitive closure of a relation

Let R be an equivalence relation. Then the following three statements are equivalent: (1) $a R b$; (2) $[a]_R \cap [b]_R \neq \emptyset$; (3) $[a]_R = [b]_R$.

The equivalence classes of an equivalence relation on a set A form a partition of A . Conversely, an equivalence relation can be constructed from any partition so that the equivalence classes are the subsets in the partition.

The principle of well-ordered induction

The topological sorting algorithm

Review Questions

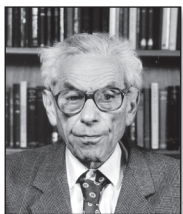
- What is a relation on a set?
 - How many relations are there on a set with n elements?
- What is a reflexive relation?
 - What is a symmetric relation?
 - What is an antisymmetric relation?
 - What is a transitive relation?
- Give an example of a relation on the set $\{1, 2, 3, 4\}$ that is
 - reflexive, symmetric, and not transitive.
 - not reflexive, symmetric, and transitive.
 - reflexive, antisymmetric, and not transitive.
 - reflexive, symmetric, and transitive.
 - reflexive, antisymmetric, and transitive.
- How many reflexive relations are there on a set with n elements?
 - How many symmetric relations are there on a set with n elements?
 - How many antisymmetric relations are there on a set with n elements?
- Explain how an n -ary relation can be used to represent information about students at a university.
 - How can the 5-ary relation containing names of students, their addresses, telephone numbers, majors, and grade point averages be used to form a 3-ary relation containing the names of students, their majors, and their grade point averages?
 - How can the 4-ary relation containing names of students, their addresses, telephone numbers, and majors and the 4-ary relation containing names of students, their student numbers, majors, and numbers of credit hours be combined into a single n -ary relation?
- Explain how to use a zero-one matrix to represent a relation on a finite set.
 - Explain how to use the zero-one matrix representing a relation to determine whether the relation is reflexive, symmetric, and/or antisymmetric.
- Explain how to use a directed graph to represent a relation on a finite set.
 - Explain how to use the directed graph representing a relation to determine whether a relation is reflexive, symmetric, and/or antisymmetric.
- Define the reflexive closure and the symmetric closure of a relation.
 - How can you construct the reflexive closure of a relation?
 - How can you construct the symmetric closure of a relation?
 - Find the reflexive closure and the symmetric closure of the relation $\{(1, 2), (2, 3), (2, 4), (3, 1)\}$ on the set $\{1, 2, 3, 4\}$.
- Define the transitive closure of a relation.
 - Can the transitive closure of a relation be obtained by including all pairs (a, c) such that (a, b) and (b, c) belong to the relation?

- c) Describe two algorithms for finding the transitive closure of a relation.
- d) Find the transitive closure of the relation $\{(1,1), (1,3), (2,1), (2,3), (2,4), (3,2), (3,4), (4,1)\}$.
10. a) Define an equivalence relation.
- b) Which relations on the set $\{a, b, c, d\}$ are equivalence relations and contain (a, b) and (b, d) ?
11. a) Show that congruence modulo m is an equivalence relation whenever m is a positive integer.
- b) Show that the relation $\{(a, b) \mid a \equiv \pm b \pmod{7}\}$ is an equivalence relation on the set of integers.
12. a) What are the equivalence classes of an equivalence relation?
- b) What are the equivalence classes of the “congruent modulo 5” relation?
- c) What are the equivalence classes of the equivalence relation in Question 11(b)?
13. Explain the relationship between equivalence relations on a set and partitions of this set.
14. a) Define a partial ordering.
- b) Show that the divisibility relation on the set of positive integers is a partial order.
15. Explain how partial orderings on the sets A_1 and A_2 can be used to define a partial ordering on the set $A_1 \times A_2$.
16. a) Explain how to construct the Hasse diagram of a partial order on a finite set.
- b) Draw the Hasse diagram of the divisibility relation on the set $\{2, 3, 5, 9, 12, 15, 18\}$.
17. a) Define a maximal element of a poset and the greatest element of a poset.
- b) Give an example of a poset that has three maximal elements.
- c) Give an example of a poset with a greatest element.
18. a) Define a lattice.
- b) Give an example of a poset with five elements that is a lattice and an example of a poset with five elements that is not a lattice.
19. a) Show that every finite subset of a lattice has a greatest lower bound and a least upper bound.
- b) Show that every lattice with a finite number of elements has a least element and a greatest element.
20. a) Define a well-ordered set.
- b) Describe an algorithm for producing a totally ordered set compatible with a given partially ordered set.
- c) Explain how the algorithm from (b) can be used to order the tasks in a project if tasks are done one at a time and each task can be done only after one or more of the other tasks have been completed.

Supplementary Exercises

1. Let S be the set of all strings of English letters. Determine whether these relations are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.
- a) $R_1 = \{(a, b) \mid a \text{ and } b \text{ have no letters in common}\}$
- b) $R_2 = \{(a, b) \mid a \text{ and } b \text{ are not the same length}\}$
- c) $R_3 = \{(a, b) \mid a \text{ is longer than } b\}$
2. Construct a relation on the set $\{a, b, c, d\}$ that is
- a) reflexive, symmetric, but not transitive.
- b) irreflexive, symmetric, and transitive.
- c) irreflexive, antisymmetric, and not transitive.
- d) reflexive, neither symmetric nor antisymmetric, and transitive.
- e) neither reflexive, irreflexive, symmetric, antisymmetric, nor transitive.
3. Show that the relation R on $\mathbf{Z} \times \mathbf{Z}$ defined by $(a, b) R (c, d)$ if and only if $a + d = b + c$ is an equivalence relation.
4. Show that a subset of an antisymmetric relation is also antisymmetric.
5. Let R be a reflexive relation on a set A . Show that $R \subseteq R^2$.
6. Suppose that R_1 and R_2 are reflexive relations on a set A . Show that $R_1 \oplus R_2$ is irreflexive.
7. Suppose that R_1 and R_2 are reflexive relations on a set A . Is $R_1 \cap R_2$ also reflexive? Is $R_1 \cup R_2$ also reflexive?
8. Suppose that R is a symmetric relation on a set A . Is \bar{R} also symmetric?
9. Let R_1 and R_2 be symmetric relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also symmetric?
10. A relation R is called **circular** if aRb and bRc imply that cRa . Show that R is reflexive and circular if and only if it is an equivalence relation.
11. Show that a primary key in an n -ary relation is a primary key in any projection of this relation that contains this key as one of its fields.
12. Is the primary key in an n -ary relation also a primary key in a larger relation obtained by taking the join of this relation with a second relation?
13. Show that the reflexive closure of the symmetric closure of a relation is the same as the symmetric closure of its reflexive closure.
14. Let R be the relation on the set of all mathematicians that contains the ordered pair (a, b) if and only if a and b have written a published mathematical paper together.
- a) Describe the relation R^2 .
- b) Describe the relation R^* .
- c) The **Erdős number** of a mathematician is 1 if this mathematician wrote a paper with the prolific Hungarian mathematician Paul Erdős, it is 2 if this mathematician did not write a joint paper with Erdős but wrote a joint paper with someone who wrote a joint paper with Erdős, and so on (except that the Erdős number of Erdős himself is 0). Give a definition of the Erdős number in terms of paths in R .

15. a) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.
 b) Show, however, that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.
16. a) Let S be the set of subroutines of a computer program. Define the relation R by $\mathbf{P} R \mathbf{Q}$ if subroutine \mathbf{P} calls subroutine \mathbf{Q} during its execution. Describe the transitive closure of R .
 b) For which subroutines \mathbf{P} does (\mathbf{P}, \mathbf{P}) belong to the transitive closure of R ?
 c) Describe the reflexive closure of the transitive closure of R .
17. Suppose that R and S are relations on a set A with $R \subseteq S$ such that the closures of R and S with respect to a property \mathbf{P} both exist. Show that the closure of R with respect to \mathbf{P} is a subset of the closure of S with respect to \mathbf{P} .
18. Show that the symmetric closure of the union of two relations is the union of their symmetric closures.
- *19. Devise an algorithm, based on the concept of interior vertices, that finds the length of the longest path between two vertices in a directed graph, or determines that there are arbitrarily long paths between these vertices.
20. Which of these are equivalence relations on the set of all people?
- a) $\{(x, y) \mid x \text{ and } y \text{ have the same sign of the zodiac}\}$
 b) $\{(x, y) \mid x \text{ and } y \text{ were born in the same year}\}$
 c) $\{(x, y) \mid x \text{ and } y \text{ have been in the same city}\}$
- *21. How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?
22. Show that $\{(x, y) \mid x - y \in \mathbf{Q}\}$ is an equivalence relation on the set of real numbers, where \mathbf{Q} denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?
23. Suppose that $P_1 = \{A_1, A_2, \dots, A_m\}$ and $P_2 = \{B_1, B_2, \dots, B_n\}$ are both partitions of the set S . Show that the collection of nonempty subsets of the form $A_i \cap B_j$ is a partition of S that is a refinement of both P_1 and P_2 (see the preamble to Exercise 49 of Section 9.5).
- *24. Show that the transitive closure of the symmetric closure of the reflexive closure of a relation R is the smallest equivalence relation that contains R .
25. Let $\mathbf{R}(S)$ be the set of all relations on a set S . Define the relation \preceq on $\mathbf{R}(S)$ by $R_1 \preceq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S . Show that $(\mathbf{R}(S), \preceq)$ is a poset.
26. Let $\mathbf{P}(S)$ be the set of all partitions of the set S . Define the relation \preceq on $\mathbf{P}(S)$ by $P_1 \preceq P_2$ if P_1 is a refinement of P_2 (see Exercise 49 of Section 9.5). Show that $(\mathbf{P}(S), \preceq)$ is a poset.



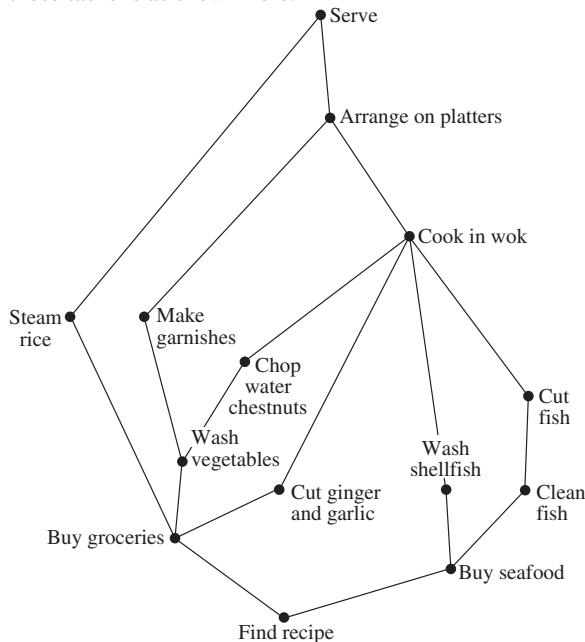
PAUL ERDŐS (1913–1996) Paul Erdős, born in Budapest, Hungary, was the son of two high school mathematics teachers. He was a child prodigy; at age 3 he could multiply three-digit numbers in his head, and at 4 he discovered negative numbers on his own. Because his mother did not want to expose him to contagious diseases, he was mostly home-schooled. At 17 Erdős entered Eötvös University, graduating four years later with a Ph.D. in mathematics. After graduating he spent four years at Manchester, England, on a postdoctoral fellowship. In 1938 he went to the United States because of the difficult political situation in Hungary, especially for Jews. He spent much of his time in the United States, except for 1954 to 1962, when he was banned as part of the paranoia of the McCarthy era. He also spent considerable time in Israel.

Erdős made many significant contributions to combinatorics and to number theory. One of the discoveries of which he was most proud is his elementary proof (in the sense that it does not use any complex analysis) of the prime number theorem, which provides an estimate for the number of primes not exceeding a fixed positive integer. He also participated in the modern development of the Ramsey theory.

Erdős traveled extensively throughout the world to work with other mathematicians, visiting conferences, universities, and research laboratories. He had no permanent home. He devoted himself almost entirely to mathematics, traveling from one mathematician to the next, proclaiming “My brain is open.” Erdős was the author or coauthor of more than 1500 papers and had more than 500 coauthors. Copies of his articles are kept by Ron Graham, a famous discrete mathematician with whom he collaborated extensively and who took care of many of his worldly needs.

Erdős offered rewards, ranging from \$10 to \$10,000, for the solution of problems that he found particularly interesting, with the size of the reward depending on the difficulty of the problem. He paid out close to \$4000. Erdős had his own special language, using such terms as “epsilon” (child), “boss” (woman), “slave” (man), “captured” (married), “liberated” (divorced), “Supreme Fascist” (God), “Sam” (United States), and “Joe” (Soviet Union). Although he was curious about many things, he concentrated almost all his energy on mathematical research. He had no hobbies and no full-time job. He never married and apparently remained celibate. Erdős was extremely generous, donating much of the money he collected from prizes, awards, and stipends for scholarships and to worthwhile causes. He traveled extremely lightly and did not like having many material possessions.

27. Schedule the tasks needed to cook a Chinese meal by specifying their order, if the Hasse diagram representing these tasks is as shown here.



A subset of a poset such that every two elements of this subset are comparable is called a **chain**. A subset of a poset is called an **antichain** if every two elements of this subset are incomparable.

28. Find all chains in the posets with the Hasse diagrams shown in Exercises 25–27 in Section 9.6.
29. Find all antichains in the posets with the Hasse diagrams shown in Exercises 25–27 in Section 9.6.
30. Find an antichain with the greatest number of elements in the poset with the Hasse diagram of Exercise 32 in Section 9.6.
31. Show that every maximal chain in a finite poset (S, \preceq) contains a minimal element of S . (A maximal chain is a chain that is not a subset of a larger chain.)
- **32.** Show that every finite poset can be partitioned into k chains, where k is the largest number of elements in an antichain in this poset.
- *33.** Show that in any group of $mn + 1$ people there is either a list of $m + 1$ people where a person in the list (except for the first person listed) is a descendant of the previous person on the list, or there are $n + 1$ people such that none of these people is a descendant of any of the other n people. [Hint: Use Exercise 32.]

Suppose that (S, \preceq) is a well-founded partially ordered set. The *principle of well-founded induction* states that $P(x)$ is true for all $x \in S$ if $\forall x (\forall y (y \prec x \rightarrow P(y)) \rightarrow P(x))$.

34. Show that no separate basis case is needed for the principle of well-founded induction. That is, $P(u)$ is true for all minimal elements u in S if $\forall x (\forall y (y \prec x \rightarrow P(y)) \rightarrow P(x))$.

- *35.** Show that the principle of well-founded induction is valid.

A relation R on a set A is a **quasi-ordering** on A if R is reflexive and transitive.

36. Let R be the relation on the set of all functions from \mathbb{Z}^+ to \mathbb{Z}^+ such that (f, g) belongs to R if and only if f is $O(g)$. Show that R is a quasi-ordering.
37. Let R be a quasi-ordering on a set A . Show that $R \cap R^{-1}$ is an equivalence relation.
- *38.** Let R be a quasi-ordering and let S be the relation on the set of equivalence classes of $R \cap R^{-1}$ such that (C, D) belongs to S , where C and D are equivalence classes of R , if and only if there are elements c of C and d of D such that (c, d) belongs to R . Show that S is a partial ordering.

Let L be a lattice. Define the **meet** (\wedge) and **join** (\vee) operations by $x \wedge y = \text{glb}(x, y)$ and $x \vee y = \text{lub}(x, y)$.

39. Show that the following properties hold for all elements x, y , and z of a lattice L .

- a) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (**commutative laws**)
- b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ (**associative laws**)
- c) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ (**absorption laws**)
- d) $x \wedge x = x$ and $x \vee x = x$ (**idempotent laws**)

40. Show that if x and y are elements of a lattice L , then $x \vee y = y$ if and only if $x \wedge y = x$.

A lattice L is **bounded** if it has both an **upper bound**, denoted by 1 , such that $x \preceq 1$ for all $x \in L$ and a **lower bound**, denoted by 0 , such that $0 \preceq x$ for all $x \in L$.

41. Show that if L is a bounded lattice with upper bound 1 and lower bound 0 then these properties hold for all elements $x \in L$.

- a) $x \vee 1 = 1$ b) $x \wedge 1 = x$
- c) $x \vee 0 = x$ d) $x \wedge 0 = 0$

42. Show that every finite lattice is bounded.

A lattice is called **distributive** if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y , and z in L .

- *43.** Give an example of a lattice that is not distributive.

44. Show that the lattice $(P(S), \subseteq)$ where $P(S)$ is the power set of a finite set S is distributive.

45. Is the lattice $(\mathbb{Z}^+, |)$ distributive?

The **complement** of an element a of a bounded lattice L with upper bound 1 and lower bound 0 is an element b such that $a \vee b = 1$ and $a \wedge b = 0$. Such a lattice is **complemented** if every element of the lattice has a complement.

46. Give an example of a finite lattice where at least one element has more than one complement and at least one element has no complement.

47. Show that the lattice $(P(S), \subseteq)$ where $P(S)$ is the power set of a finite set S is complemented.

- *48.** Show that if L is a finite distributive lattice, then an element of L has at most one complement.

The game of Chomp, introduced in Example 12 in Section 1.8, can be generalized for play on any finite partially ordered set (S, \preceq) with a least element a . In this game, a move consists of selecting an element x in S and removing x and all elements larger than it from S . The loser is the player who is forced to select the least element a .

- 49.** Show that the game of Chomp with cookies arranged in an $m \times n$ rectangular grid, described in Example 12 in Section 1.8, is the same as the game of Chomp on the poset $(S, |)$, where S is the set of all positive integers that divide $p^{m-1}q^{n-1}$, where p and q are distinct primes.
- 50.** Show that if (S, \preceq) has a greatest element b , then a winning strategy for Chomp on this poset exists. [Hint: Generalize the argument in Example 12 in Section 1.8.]

Computer Projects

Write programs with these input and output.

- Given the matrix representing a relation on a finite set, determine whether the relation is reflexive and/or irreflexive.
- Given the matrix representing a relation on a finite set, determine whether the relation is symmetric and/or anti-symmetric.
- Given the matrix representing a relation on a finite set, determine whether the relation is transitive.
- Given a positive integer n , display all the relations on a set with n elements.
- *5.** Given a positive integer n , determine the number of transitive relations on a set with n elements.
- *6.** Given a positive integer n , determine the number of equivalence relations on a set with n elements.
- *7.** Given a positive integer n , display all the equivalence relations on the set of the n smallest positive integers.
- Given an n -ary relation, find the projection of this relation when specified fields are deleted.
- Given an m -ary relation and an n -ary relation, and a set of common fields, find the join of these relations with respect to these common fields.
- Given the matrix representing a relation on a finite set, find the matrix representing the reflexive closure of this relation.
- Given the matrix representing a relation on a finite set, find the matrix representing the symmetric closure of this relation.
- Given the matrix representing a relation on a finite set, find the matrix representing the transitive closure of this relation by computing the join of the Boolean powers of the matrix representing the relation.
- Given the matrix representing a relation on a finite set, find the matrix representing the transitive closure of this relation using Warshall's algorithm.
- Given the matrix representing a relation on a finite set, find the matrix representing the smallest equivalence relation containing this relation.
- Given a partial ordering on a finite set, find a total ordering compatible with it using topological sorting.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

- Display all the different relations on a set with four elements.
- Display all the different reflexive and symmetric relations on a set with six elements.
- Display all the reflexive and transitive relations on a set with five elements.
- *4.** Determine how many transitive relations there are on a set with n elements for all positive integers n with $n \leq 7$.
- Find the transitive closure of a relation of your choice on a set with at least 20 elements. Either use a relation that corresponds to direct links in a particular transportation or communications network or use a randomly generated relation.
- Compute the number of different equivalence relations on a set with n elements for all positive integers n not exceeding 20.
- Display all the equivalence relations on a set with seven elements.
- *8.** Display all the partial orders on a set with five elements.
- *9.** Display all the lattices on a set with five elements.