

10

Graphs

10.1 Graphs and Graph Models

10.2 Graph Terminology and Special Types of Graphs

10.3 Representing Graphs and Graph Isomorphism

10.4 Connectivity

10.5 Euler and Hamilton Paths

10.6 Shortest-Path Problems

10.7 Planar Graphs

10.8 Graph Coloring

Graphs are discrete structures consisting of vertices and edges that connect these vertices. There are different kinds of graphs, depending on whether edges have directions, whether multiple edges can connect the same pair of vertices, and whether loops are allowed. Problems in almost every conceivable discipline can be solved using graph models. We will give examples to illustrate how graphs are used as models in a variety of areas. For instance, we will show how graphs are used to represent the competition of different species in an ecological niche, how graphs are used to represent who influences whom in an organization, and how graphs are used to represent the outcomes of round-robin tournaments. We will describe how graphs can be used to model acquaintanceships between people, collaboration between researchers, telephone calls between telephone numbers, and links between websites. We will show how graphs can be used to model roadmaps and the assignment of jobs to employees of an organization.

Using graph models, we can determine whether it is possible to walk down all the streets in a city without going down a street twice, and we can find the number of colors needed to color the regions of a map. Graphs can be used to determine whether a circuit can be implemented on a planar circuit board. We can distinguish between two chemical compounds with the same molecular formula but different structures using graphs. We can determine whether two computers are connected by a communications link using graph models of computer networks. Graphs with weights assigned to their edges can be used to solve problems such as finding the shortest path between two cities in a transportation network. We can also use graphs to schedule exams and assign channels to television stations. This chapter will introduce the basic concepts of graph theory and present many different graph models. To solve the wide variety of problems that can be studied using graphs, we will introduce many different graph algorithms. We will also study the complexity of these algorithms.

10.1 Graphs and Graph Models

We begin with the definition of a graph.

DEFINITION 1

A graph $G = (V, E)$ consists of V , a nonempty set of *vertices* (or *nodes*) and E , a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

Remark: The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an **infinite graph**, and in comparison, a graph with a finite vertex set and a finite edge set is called a **finite graph**. In this book we will usually consider only finite graphs.

Now suppose that a network is made up of data centers and communication links between computers. We can represent the location of each data center by a point and each communications link by a line segment, as shown in Figure 1.

This computer network can be modeled using a graph in which the vertices of the graph represent the data centers and the edges represent communication links. In general, we visualize

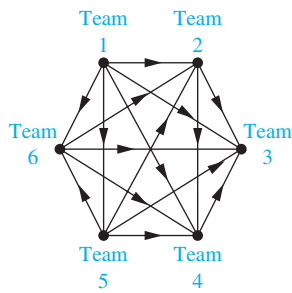


FIGURE 13 A Graph Model of a Round-Robin Tournament.

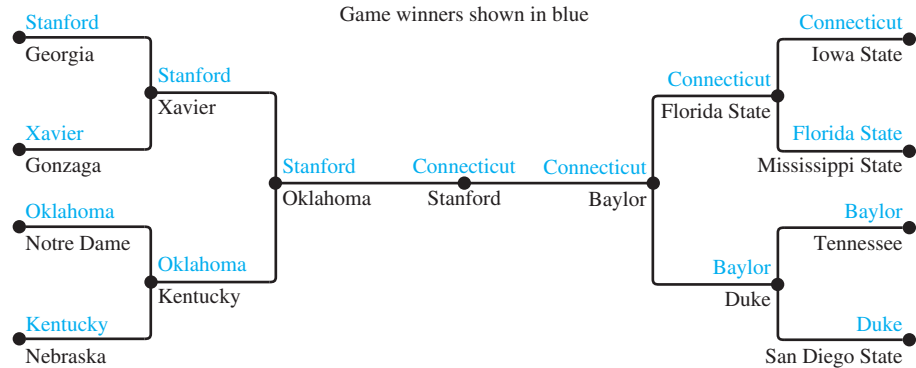


FIGURE 14 A Single-Elimination Tournament.

TOURNAMENTS We now give some examples that show how graphs can also be used to model different kinds of tournaments.

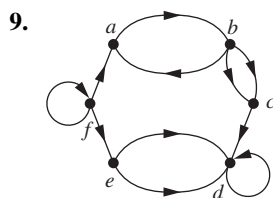
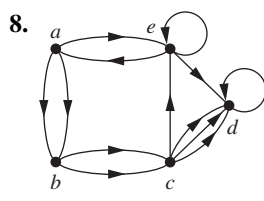
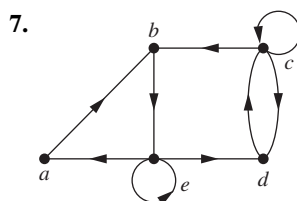
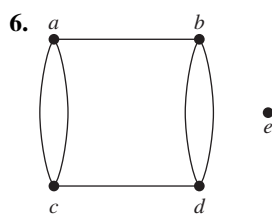
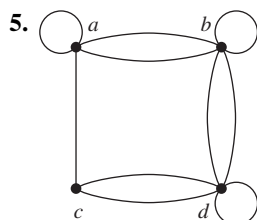
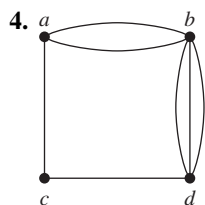
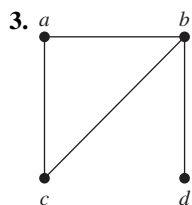
EXAMPLE 13 Round-Robin Tournaments A tournament where each team plays every other team exactly once and no ties are allowed is called a **round-robin tournament**. Such tournaments can be modeled using directed graphs where each team is represented by a vertex. Note that (a, b) is an edge if team a beats team b . This graph is a simple directed graph, containing no loops or multiple directed edges (because no two teams play each other more than once). Such a directed graph model is presented in Figure 13. We see that Team 1 is undefeated in this tournament, and Team 3 is winless.

EXAMPLE 14 Single-Elimination Tournaments A tournament where each contestant is eliminated after one loss is called a **single-elimination tournament**. Single-elimination tournaments are often used in sports, including tennis championships and the yearly NCAA basketball championship. We can model such a tournament using a vertex to represent each game and a directed edge to connect a game to the next game the winner of this game played in. The graph in Figure 14 represents the games played by the final 16 teams in the 2010 NCAA women's basketball tournament.

Exercises

- Draw graph models, stating the type of graph (from Table 1) used, to represent airline routes where every day there are four flights from Boston to Newark, two flights from Newark to Boston, three flights from Newark to Miami, two flights from Miami to Newark, one flight from Newark to Detroit, two flights from Detroit to Newark, three flights from Newark to Washington, two flights from Washington to Newark, and one flight from Washington to Miami, with
 - an edge between vertices representing cities that have a flight between them (in either direction).
 - an edge between vertices representing cities for each flight that operates between them (in either direction).
 - an edge between vertices representing cities for each flight that operates between them (in either direction), plus a loop for a special sightseeing trip that takes off and lands in Miami.
 - an edge from a vertex representing a city where a flight starts to the vertex representing the city where it ends.
 - an edge for each flight from a vertex representing a city where the flight begins to the vertex representing the city where the flight ends.
- What kind of graph (from Table 1) can be used to model a highway system between major cities where
 - there is an edge between the vertices representing cities if there is an interstate highway between them?
 - there is an edge between the vertices representing cities for each interstate highway between them?
 - there is an edge between the vertices representing cities for each interstate highway between them, and there is a loop at the vertex representing a city if there is an interstate highway that circles this city?

For Exercises 3–9, determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops. Use your answers to determine the type of graph in Table 1 this graph is.



10. For each undirected graph in Exercises 3–9 that is not simple, find a set of edges to remove to make it simple.

11. Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G .

12. Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on G .

13. The **intersection graph** of a collection of sets A_1, A_2, \dots, A_n is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets.

a) $A_1 = \{0, 2, 4, 6, 8\}$, $A_2 = \{0, 1, 2, 3, 4\}$,
 $A_3 = \{1, 3, 5, 7, 9\}$, $A_4 = \{5, 6, 7, 8, 9\}$,
 $A_5 = \{0, 1, 8, 9\}$

b) $A_1 = \{\dots, -4, -3, -2, -1, 0\}$,
 $A_2 = \{\dots, -2, -1, 0, 1, 2, \dots\}$,
 $A_3 = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$,
 $A_4 = \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$,
 $A_5 = \{\dots, -6, -3, 0, 3, 6, \dots\}$

c) $A_1 = \{x \mid x < 0\}$,
 $A_2 = \{x \mid -1 < x < 0\}$,
 $A_3 = \{x \mid 0 < x < 1\}$,
 $A_4 = \{x \mid -1 < x < 1\}$,
 $A_5 = \{x \mid x > -1\}$,
 $A_6 = \mathbf{R}$

14. Use the niche overlap graph in Figure 11 to determine the species that compete with hawks.

15. Construct a niche overlap graph for six species of birds, where the hermit thrush competes with the robin and with the blue jay, the robin also competes with the mockingbird, the mockingbird also competes with the blue jay, and the nuthatch competes with the hairy woodpecker.

16. Draw the acquaintanceship graph that represents that Tom and Patricia, Tom and Hope, Tom and Sandy, Tom and Amy, Tom and Marika, Jeff and Patricia, Jeff and Mary, Patricia and Hope, Amy and Hope, and Amy and Marika know each other, but none of the other pairs of people listed know each other.

17. We can use a graph to represent whether two people were alive at the same time. Draw such a graph to represent whether each pair of the mathematicians and computer scientists with biographies in the first five chapters of this book who died before 1900 were contemporaneous. (Assume two people lived at the same time if they were alive during the same year.)

18. Who can influence Fred and whom can Fred influence in the influence graph in Example 2?

19. Construct an influence graph for the board members of a company if the President can influence the Director of Research and Development, the Director of Marketing, and the Director of Operations; the Director of Research and Development can influence the Director of Operations; the Director of Marketing can influence the Director of Operations; and no one can influence, or be influenced by, the Chief Financial Officer.

20. Which other teams did Team 4 beat and which teams beat Team 4 in the round-robin tournament represented by the graph in Figure 13?

21. In a round-robin tournament the Tigers beat the Blue Jays, the Tigers beat the Cardinals, the Tigers beat the Orioles, the Blue Jays beat the Cardinals, the Blue Jays beat the Orioles, and the Cardinals beat the Orioles. Model this outcome with a directed graph.

22. Construct the call graph for a set of seven telephone numbers 555-0011, 555-1221, 555-1333, 555-8888, 555-2222, 555-0091, and 555-1200 if there were three calls from 555-0011 to 555-8888 and two calls from 555-8888 to 555-0011, two calls from 555-2222 to 555-0091, two calls from 555-1221 to each of the other numbers, and one call from 555-1333 to each of 555-0011, 555-1221, and 555-1200.

23. Explain how the two telephone call graphs for calls made during the month of January and calls made during the month of February can be used to determine the new telephone numbers of people who have changed their telephone numbers.

24. a) Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?
 b) Describe a graph that models the electronic mail sent in a network in a particular week.
25. How can a graph that models e-mail messages sent in a network be used to find people who have recently changed their primary e-mail address?
26. How can a graph that models e-mail messages sent in a network be used to find electronic mail mailing lists used to send the same message to many different e-mail addresses?
27. Describe a graph model that represents whether each person at a party knows the name of each other person at the party. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?
28. Describe a graph model that represents a subway system in a large city. Should edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?
29. For each course at a university, there may be one or more other courses that are its prerequisites. How can a graph be used to model these courses and which courses are prerequisites for which courses? Should edges be directed or undirected? Looking at the graph model, how can we find courses that do not have any prerequisites and how can we find courses that are not the prerequisite for any other courses?
30. Describe a graph model that represents the positive recommendations of movie critics, using vertices to represent both these critics and all movies that are currently being shown.
31. Describe a graph model that represents traditional marriages between men and women. Does this graph have any special properties?
32. Which statements must be executed before S_6 is executed in the program in Example 8? (Use the precedence graph in Figure 10.)
33. Construct a precedence graph for the following program:
- $$\begin{aligned} S_1: x &:= 0 \\ S_2: x &:= x + 1 \\ S_3: y &:= 2 \\ S_4: z &:= y \\ S_5: x &:= x + 2 \\ S_6: y &:= x + z \\ S_7: z &:= 4 \end{aligned}$$
34. Describe a discrete structure based on a graph that can be used to model airline routes and their flight times. [Hint: Add structure to a directed graph.]
35. Describe a discrete structure based on a graph that can be used to model relationships between pairs of individuals in a group, where each individual may either like, dislike, or be neutral about another individual, and the reverse relationship may be different. [Hint: Add structure to a directed graph. Treat separately the edges in opposite directions between vertices representing two individuals.]
36. Describe a graph model that can be used to represent all forms of electronic communication between two people in a single graph. What kind of graph is needed?

10.2 Graph Terminology and Special Types of Graphs

Introduction



We introduce some of the basic vocabulary of graph theory in this section. We will use this vocabulary later in this chapter when we solve many different types of problems. One such problem involves determining whether a graph can be drawn in the plane so that no two of its edges cross. Another example is deciding whether there is a one-to-one correspondence between the vertices of two graphs that produces a one-to-one correspondence between the edges of the graphs. We will also introduce several important families of graphs often used as examples and in models. Several important applications will be described where these special types of graphs arise.

Basic Terminology

First, we give some terminology that describes the vertices and edges of undirected graphs.

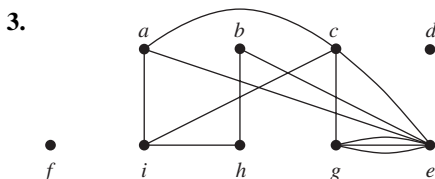
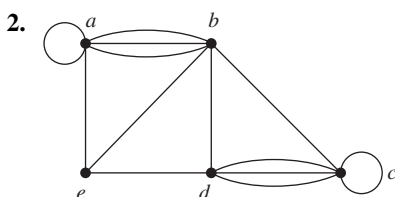
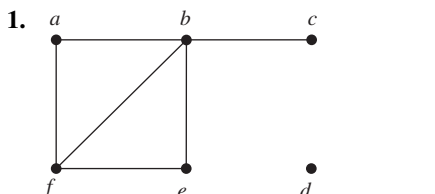
DEFINITION 1

Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G . Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v .

Solution: The vertex set of the union $G_1 \cup G_2$ is the union of the two vertex sets, namely, $\{a, b, c, d, e, f\}$. The edge set of the union is the union of the two edge sets. The union is displayed in Figure 16(b).

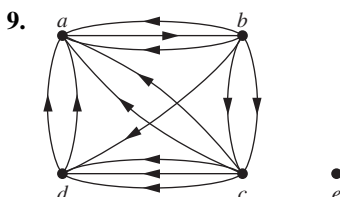
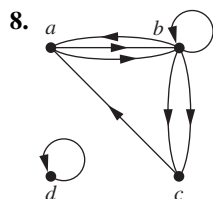
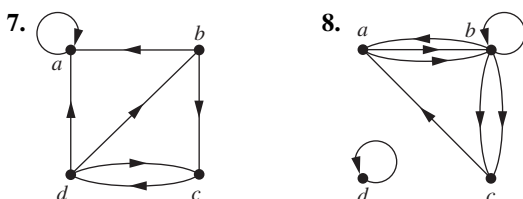
Exercises

In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.



4. Find the sum of the degrees of the vertices of each graph in Exercises 1–3 and verify that it equals twice the number of edges in the graph.
5. Can a simple graph exist with 15 vertices each of degree five?
6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

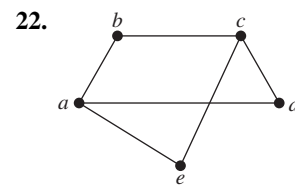
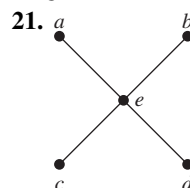
In Exercises 7–9 determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.



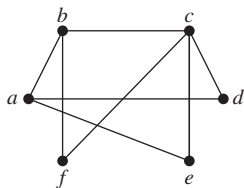
10. For each of the graphs in Exercises 7–9 determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.
11. Construct the underlying undirected graph for the graph with directed edges in Figure 2.
12. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What does the neighborhood a vertex in this graph represent? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?
13. What does the degree of a vertex represent in an academic collaboration graph? What does the neighborhood of a vertex represent? What do isolated and pendant vertices represent?
14. What does the degree of a vertex in the Hollywood graph represent? What does the neighborhood of a vertex represent? What do the isolated and pendant vertices represent?
15. What do the in-degree and the out-degree of a vertex in a telephone call graph, as described in Example 4 of Section 10.1, represent? What does the degree of a vertex in the undirected version of this graph represent?
16. What do the in-degree and the out-degree of a vertex in the Web graph, as described in Example 5 of Section 10.1, represent?
17. What do the in-degree and the out-degree of a vertex in a directed graph modeling a round-robin tournament represent?
18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.
19. Use Exercise 18 to show that in a group of people, there must be two people who are friends with the same number of other people in the group.
20. Draw these graphs.

a) K_7	b) $K_{1,8}$	c) $K_{4,4}$
d) C_7	e) W_7	f) Q_4

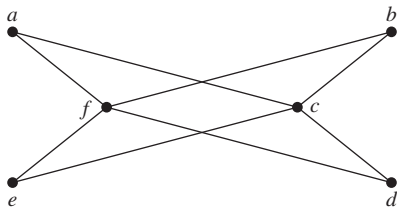
In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.



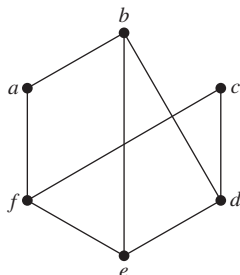
23.



24.



25.

26. For which values of n are these graphs bipartite?

- a) K_n b) C_n c) W_n d) Q_n

27. Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

- Use a bipartite graph to model the four employees and their qualifications.
- Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.
- If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

28. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsibilities: planning, publicity, sales, marketing, development, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development; Smith could do publicity, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations.

- Model the capabilities of these employees using a bipartite graph.
- Find an assignment of responsibilities such that each employee is assigned one responsibility.

c) Is the matching of responsibilities you found in part (b) a complete matching? Is it a maximum matching?

29. Suppose that there are five young women and five young men on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda. Use Hall's theorem to show there is no matching of the young men and young women on the island such that each young man is matched with a young woman he is willing to marry.

30. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

- Model the possible marriages on the island using a bipartite graph.
- Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.
- Is the matching you found in part (b) a complete matching? Is it a maximum matching?

*31. Suppose there is an integer k such that every man on a desert island is willing to marry exactly k of the women on the island and every woman on the island is willing to marry exactly k of the men. Also, suppose that a man is willing to marry a woman if and only if she is willing to marry him. Show that it is possible to match the men and women on the island so that everyone is matched with someone that they are willing to marry.

*32. In this exercise we prove a theorem of Øystein Ore. Suppose that $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2) and that $A \subseteq V_1$. Show that the maximum number of vertices of V_1 that are the endpoints of a matching of G equals $|V_1| - \max_{A \subseteq V_1} \text{def}(A)$, where $\text{def}(A) = |A| - |N(A)|$. (Here, $\text{def}(A)$ is called the **deficiency** of A .) [Hint: Form a larger graph by adding $\max_{A \subseteq V_1} \text{def}(A)$ new vertices to V_2 and connect all of them to the vertices of V_1 .]

33. For the graph G in Exercise 1 find

- the subgraph induced by the vertices a, b, c , and f .
- the new graph G_1 obtained from G by contracting the edge connecting b and f .

34. Let n be a positive integer. Show that a subgraph induced by a nonempty subset of the vertex set of K_n is a complete graph.

35. How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n
 d) $K_{m,n}$ e) Q_n

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 is 4, 4, 4, 3, 2, 1, 0.

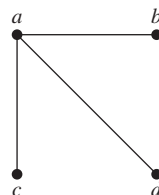
36. Find the degree sequences for each of the graphs in Exercises 21–25.
 37. Find the degree sequence of each of the following graphs.
 a) K_4 b) C_4 c) W_4
 d) $K_{2,3}$ e) Q_3
 38. What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.
 39. What is the degree sequence of K_n , where n is a positive integer? Explain your answer.
 40. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.
 41. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

A sequence d_1, d_2, \dots, d_n is called **graphic** if it is the degree sequence of a simple graph.

42. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.
 a) 5, 4, 3, 2, 1, 0 b) 6, 5, 4, 3, 2, 1 c) 2, 2, 2, 2, 2, 2
 d) 3, 3, 3, 2, 2, 2 e) 3, 3, 2, 2, 2, 2 f) 1, 1, 1, 1, 1, 1
 g) 5, 3, 3, 3, 3, 3 h) 5, 5, 4, 3, 2, 1
 43. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.
 a) 3, 3, 3, 3, 2 b) 5, 4, 3, 2, 1 c) 4, 4, 3, 2, 1
 d) 4, 4, 3, 3, 3 e) 3, 2, 2, 1, 0 f) 1, 1, 1, 1, 1

- *44. Suppose that d_1, d_2, \dots, d_n is a graphic sequence. Show that there is a simple graph with vertices v_1, v_2, \dots, v_n such that $\deg(v_i) = d_i$ for $i = 1, 2, \dots, n$ and v_1 is adjacent to v_2, \dots, v_{d_1+1} .
 *45. Show that a sequence d_1, d_2, \dots, d_n of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.
 *46. Use Exercise 45 to construct a recursive algorithm for determining whether a nonincreasing sequence of positive integers is graphic.
 47. Show that every nonincreasing sequence of nonnegative integers with an even sum of its terms is the degree sequence of a pseudograph, that is, an undirected graph where loops are allowed. [Hint: Construct such a graph by first adding as many loops as possible at each vertex. Then add additional edges connecting vertices of odd degree. Explain why this construction works.]
 48. How many subgraphs with at least one vertex does K_2 have?

49. How many subgraphs with at least one vertex does K_3 have?
 50. How many subgraphs with at least one vertex does W_3 have?
 51. Draw all subgraphs of this graph.



52. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G , and let m be the minimum degree of the vertices of G . Show that
 a) $2e/v \geq m$. b) $2e/v \leq M$.

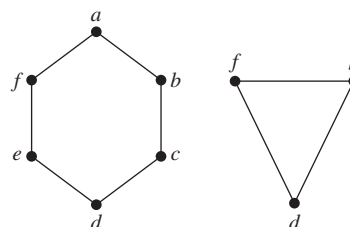
A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called **n -regular** if every vertex in this graph has degree n .

53. For which values of n are these graphs regular?
 a) K_n b) C_n c) W_n d) Q_n
 54. For which values of m and n is $K_{m,n}$ regular?

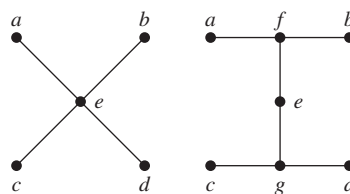
55. How many vertices does a regular graph of degree four with 10 edges have?

In Exercises 56–58 find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

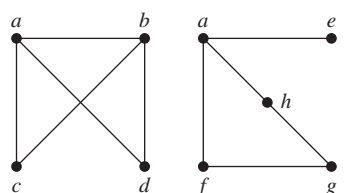
56.



57.



58.



59. The **complementary graph** \overline{G} of a simple graph G has the same vertices as G . Two vertices are adjacent in \overline{G} if and only if they are not adjacent in G . Describe each of these graphs.
 a) $\overline{K_n}$ b) $\overline{K_{m,n}}$ c) $\overline{C_n}$ d) $\overline{Q_n}$
 60. If G is a simple graph with 15 edges and \overline{G} has 13 edges, how many vertices does G have?

61. If the simple graph G has v vertices and e edges, how many edges does \overline{G} have?
62. If the degree sequence of the simple graph G is 4, 3, 3, 2, 2, what is the degree sequence of \overline{G} ?
63. If the degree sequence of the simple graph G is d_1, d_2, \dots, d_n , what is the degree sequence of \overline{G} ?
- *64. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.
65. Show that if G is a simple graph with n vertices, then the union of G and \overline{G} is K_n .
- *66. Describe an algorithm to decide whether a graph is bipartite based on the fact that a graph is bipartite if and only if it is possible to color its vertices two different colors so that no two vertices of the same color are adjacent.
- The **converse** of a directed graph $G = (V, E)$, denoted by G^{conv} , is the directed graph (V, F) , where the set F of edges of G^{conv} is obtained by reversing the direction of each edge in E .
67. Draw the converse of each of the graphs in Exercises 7–9 in Section 10.1.

68. Show that $(G^{conv})^{conv} = G$ whenever G is a directed graph.
69. Show that the graph G is its own converse if and only if the relation associated with G (see Section 9.3) is symmetric.
70. Show that if a bipartite graph $G = (V, E)$ is n -regular for some positive integer n (see the preamble to Exercise 53) and (V_1, V_2) is a bipartition of V , then $|V_1| = |V_2|$. That is, show that the two sets in a bipartition of the vertex set of an n -regular graph must contain the same number of vertices.
71. Draw the mesh network for interconnecting nine parallel processors.
72. In a variant of a mesh network for interconnecting $n = m^2$ processors, processor $P(i, j)$ is connected to the four processors $P((i \pm 1) \bmod m, j)$ and $P(i, (j \pm 1) \bmod m)$, so that connections wrap around the edges of the mesh. Draw this variant of the mesh network for 16 processors.
73. Show that every pair of processors in a mesh network of $n = m^2$ processors can communicate using $O(\sqrt{n}) = O(m)$ hops between directly connected processors.

10.3 Representing Graphs and Graph Isomorphism

Introduction

There are many useful ways to represent graphs. As we will see throughout this chapter, in working with a graph it is helpful to be able to choose its most convenient representation. In this section we will show how to represent graphs in several different ways.

Sometimes, two graphs have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs are **isomorphic**. Determining whether two graphs are isomorphic is an important problem of graph theory that we will study in this section.

Representing Graphs

One way to represent a graph without multiple edges is to list all the edges of this graph. Another way to represent a graph with no multiple edges is to use **adjacency lists**, which specify the vertices that are adjacent to each vertex of the graph.

EXAMPLE 1 Use adjacency lists to describe the simple graph given in Figure 1.

Solution: Table 1 lists those vertices adjacent to each of the vertices of the graph. ◀

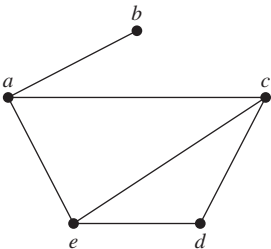


FIGURE 1 A Simple Graph.

TABLE 1 An Adjacency List for a Simple Graph.

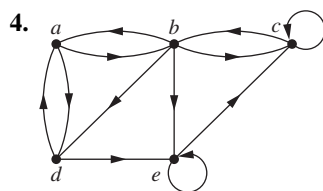
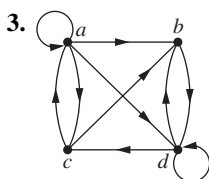
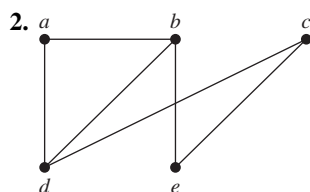
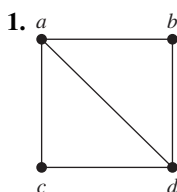
Vertex	Adjacent Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Chemists use multigraphs, known as molecular graphs, to model chemical compounds. In these graphs, vertices represent atoms and edges represent chemical bonds between these atoms. Two structural isomers, molecules with identical molecular formulas but with atoms bonded differently, have nonisomorphic molecular graphs. When a potentially new chemical compound is synthesized, a database of molecular graphs is checked to see whether the molecular graph of the compound is the same as one already known.

Electronic circuits are modeled using graphs in which vertices represent components and edges represent connections between them. Modern integrated circuits, known as chips, are miniaturized electronic circuits, often with millions of transistors and connections between them. Because of the complexity of modern chips, automation tools are used to design them. Graph isomorphism is the basis for the verification that a particular layout of a circuit produced by an automated tool corresponds to the original schematic of the design. Graph isomorphism can also be used to determine whether a chip from one vendor includes intellectual property from a different vendor. This can be done by looking for large isomorphic subgraphs in the graphs modeling these chips.

Exercises

In Exercises 1–4 use an adjacency list to represent the given graph.



5. Represent the graph in Exercise 1 with an adjacency matrix.

6. Represent the graph in Exercise 2 with an adjacency matrix.

7. Represent the graph in Exercise 3 with an adjacency matrix.

8. Represent the graph in Exercise 4 with an adjacency matrix.

9. Represent each of these graphs with an adjacency matrix.

- a) K_4 b) $K_{1,4}$ c) $K_{2,3}$
d) C_4 e) W_4 f) Q_3

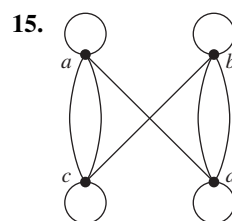
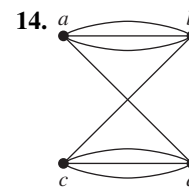
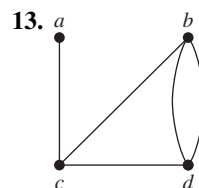
In Exercises 10–12 draw a graph with the given adjacency matrix.

10.
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

11.
$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

12.
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

In Exercises 13–15 represent the given graph using an adjacency matrix.



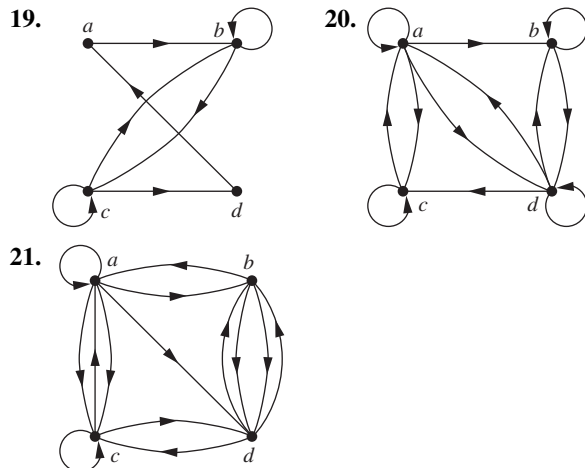
In Exercises 16–18 draw an undirected graph represented by the given adjacency matrix.

16.
$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$$

17.
$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

18.
$$\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

In Exercises 19–21 find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

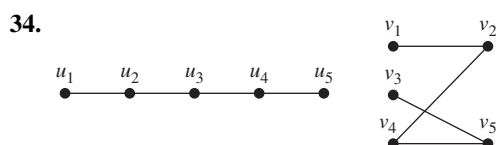


In Exercises 22–24 draw the graph represented by the given adjacency matrix.

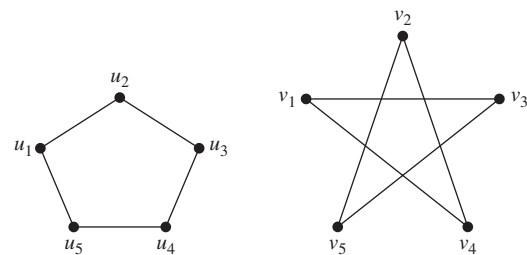
22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ 23. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ 24. $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

25. Is every zero–one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?
26. Use an incidence matrix to represent the graphs in Exercises 1 and 2.
27. Use an incidence matrix to represent the graphs in Exercises 13–15.
- *28. What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?
- *29. What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?
30. What is the sum of the entries in a row of the incidence matrix for an undirected graph?
31. What is the sum of the entries in a column of the incidence matrix for an undirected graph?
- *32. Find an adjacency matrix for each of these graphs.
a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n
- *33. Find incidence matrices for the graphs in parts (a)–(d) of Exercise 32.

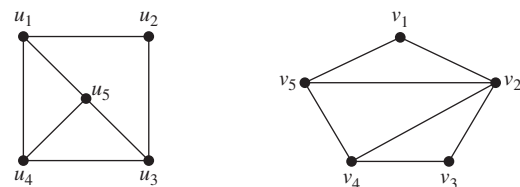
In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



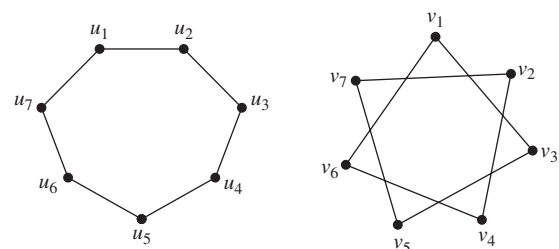
35.



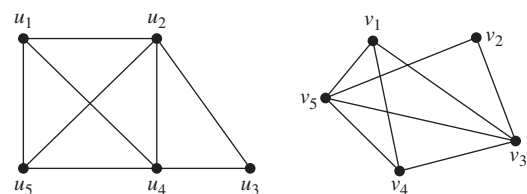
36.



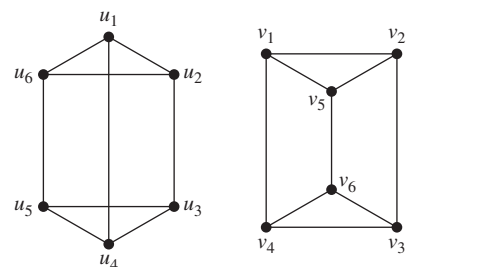
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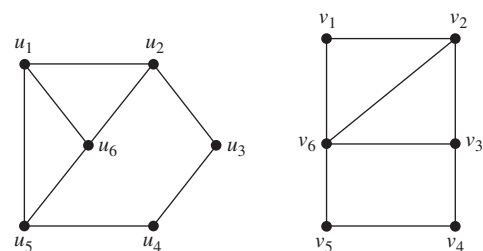
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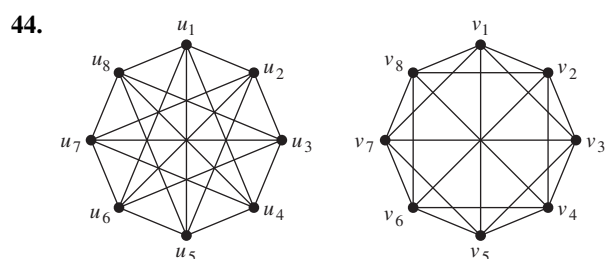
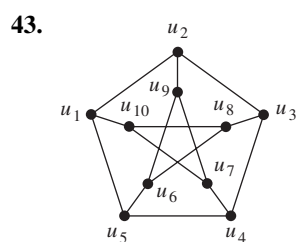
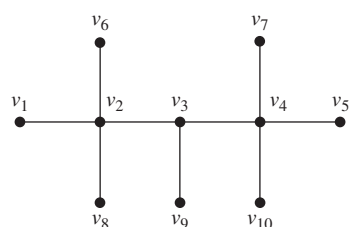
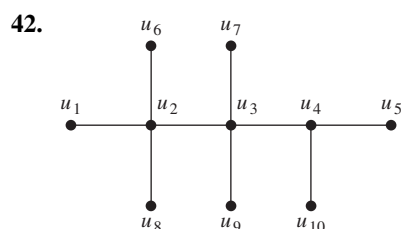
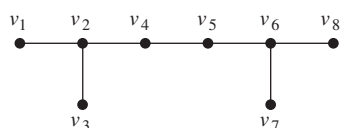
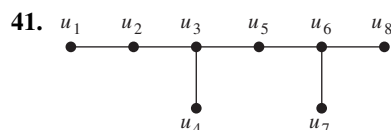


39.



40.





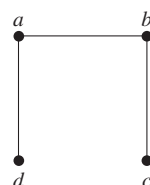
45. Show that isomorphism of simple graphs is an equivalence relation.
46. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs \overline{G} and \overline{H} are also isomorphic.
47. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex.
48. Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.
49. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix

has the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix},$$

where the four entries shown are rectangular blocks. A simple graph G is called **self-complementary** if G and \overline{G} are isomorphic.

50. Show that this graph is self-complementary.



51. Find a self-complementary simple graph with five vertices.
- *52. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod{4}$.
53. For which integers n is C_n self-complementary?
54. How many nonisomorphic simple graphs are there with n vertices, when n is
- a) 2? b) 3? c) 4?
55. How many nonisomorphic simple graphs are there with five vertices and three edges?
56. How many nonisomorphic simple graphs are there with six vertices and four edges?
57. Are the simple graphs with the following adjacency matrices isomorphic?

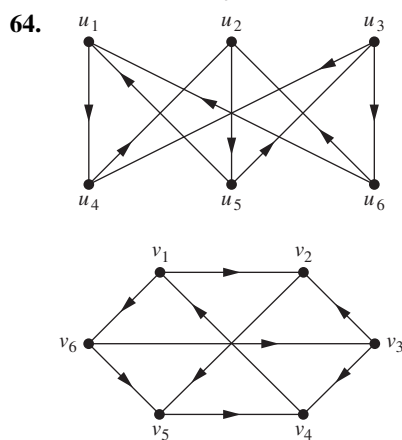
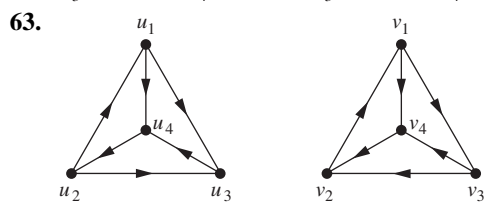
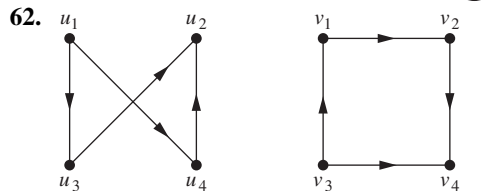
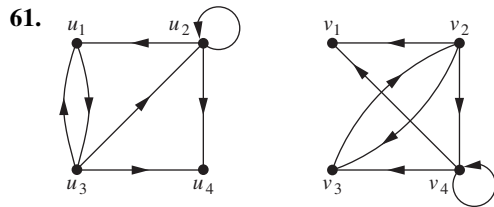
a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

58. Determine whether the graphs without loops with these incidence matrices are isomorphic.
- a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
- b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$
59. Extend the definition of isomorphism of simple graphs to undirected graphs containing loops and multiple edges.
60. Define isomorphism of directed graphs.

In Exercises 61–64 determine whether the given pair of directed graphs are isomorphic. (See Exercise 60.)



65. Show that if G and H are isomorphic directed graphs, then the converses of G and H (defined in the preamble of Exercise 67 of Section 10.2) are also isomorphic.

66. Show that the property that a graph is bipartite is an isomorphic invariant.

67. Find a pair of nonisomorphic graphs with the same degree sequence (defined in the preamble to Exercise 36 in Section 10.2) such that one graph is bipartite, but the other graph is not bipartite.

*68. How many nonisomorphic directed simple graphs are there with n vertices, when n is

- a) 2? b) 3? c) 4?

*69. What is the product of the incidence matrix and its transpose for an undirected graph?

*70. How much storage is needed to represent a simple graph with n vertices and m edges using

- a) adjacency lists?
b) an adjacency matrix?
c) an incidence matrix?

A **devil's pair** for a purported isomorphism test is a pair of nonisomorphic graphs that the test fails to show that they are not isomorphic.

71. Find a devil's pair for the test that checks the degree sequence (defined in the preamble to Exercise 36 in Section 10.2) in two graphs to make sure they agree.

72. Suppose that the function f from V_1 to V_2 is an isomorphism of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Show that it is possible to verify this fact in time polynomial in terms of the number of vertices of the graph, in terms of the number of comparisons needed.

10.4 Connectivity

Introduction

Many problems can be modeled with paths formed by traveling along the edges of graphs. For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model. Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on can be solved using models that involve paths in graphs.

Paths

Informally, a **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these edges.



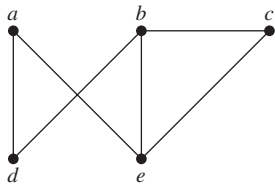
there are exactly eight paths of length four from a to d . By inspection of the graph, we see that a, b, a, b, d ; a, b, a, c, d ; a, b, d, b, d ; a, b, d, c, d ; a, c, a, b, d ; a, c, a, c, d ; a, c, d, b, d ; and a, c, d, c, d are the eight paths of length four from a to d .

Theorem 2 can be used to find the length of the shortest path between two vertices of a graph (see Exercise 56), and it can also be used to determine whether a graph is connected (see Exercises 61 and 62).

Exercises

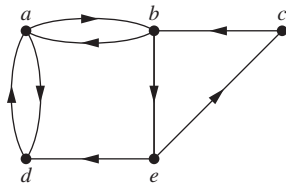
1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

- a) a, e, b, c, b b) a, e, a, d, b, c, a
c) e, b, a, d, b, e d) c, b, d, a, e, c

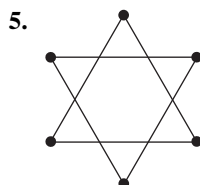


2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

- a) a, b, e, c, b b) a, d, a, d, a
c) a, d, b, e, a d) a, b, e, c, b, d, a



In Exercises 3–5 determine whether the given graph is connected.



6. How many connected components does each of the graphs in Exercises 3–5 have? For each graph find each of its connected components.

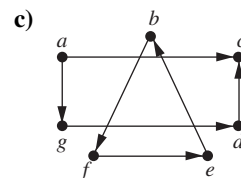
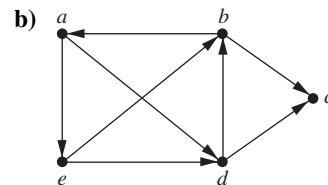
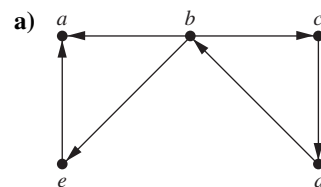
7. What do the connected components of acquaintanceship graphs represent?

8. What do the connected components of a collaboration graph represent?

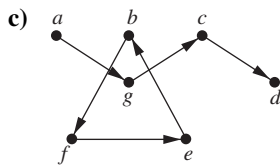
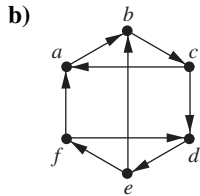
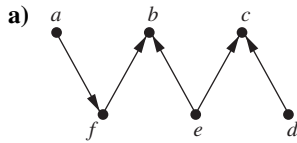
9. Explain why in the collaboration graph of mathematicians (see Example 3 in Section 10.1) a vertex representing a mathematician is in the same connected component as the vertex representing Paul Erdős if and only if that mathematician has a finite Erdős number.

10. In the Hollywood graph (see Example 3 in Section 10.1), when is the vertex representing an actor in the same connected component as the vertex representing Kevin Bacon?

11. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

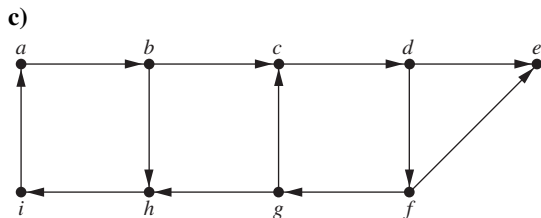
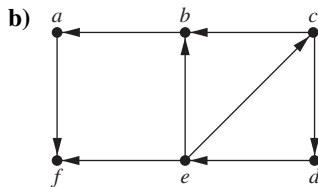
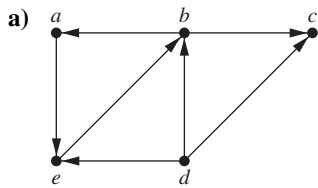


12. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

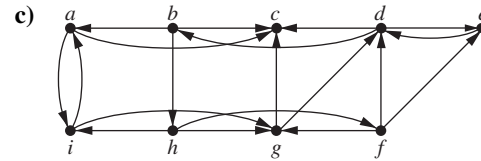
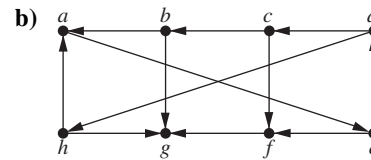
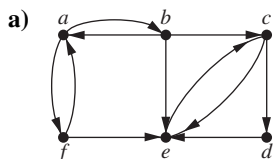


13. What do the strongly connected components of a telephone call graph represent?

14. Find the strongly connected components of each of these graphs.



15. Find the strongly connected components of each of these graphs.



Suppose that $G = (V, E)$ is a directed graph. A vertex $w \in V$ is **reachable** from a vertex $v \in V$ if there is a directed path from v to w . The vertices v and w are **mutually reachable** if there are both a directed path from v to w and a directed path from w to v in G .

16. Show that if $G = (V, E)$ is a directed graph and u, v , and w are vertices in V for which u and v are mutually reachable and v and w are mutually reachable, then u and w are mutually reachable.

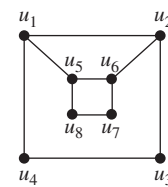
17. Show that if $G = (V, E)$ is a directed graph, then the strong components of two vertices u and v of V are either the same or disjoint. [Hint: Use Exercise 16.]

18. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in this strongly connected component.

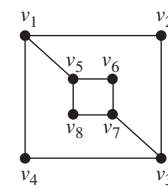
19. Find the number of paths of length n between two different vertices in K_4 if n is

a) 2. b) 3. c) 4. d) 5.

20. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

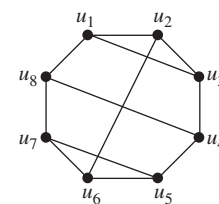


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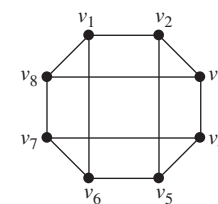


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21. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.

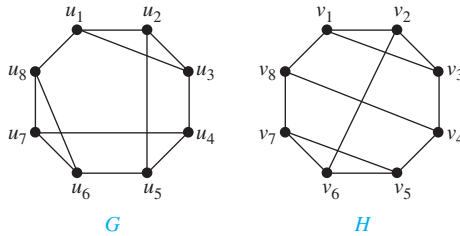


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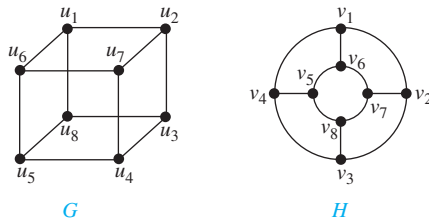


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22. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.

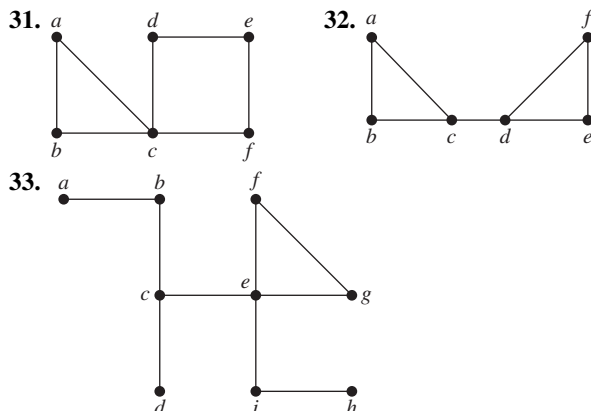


23. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



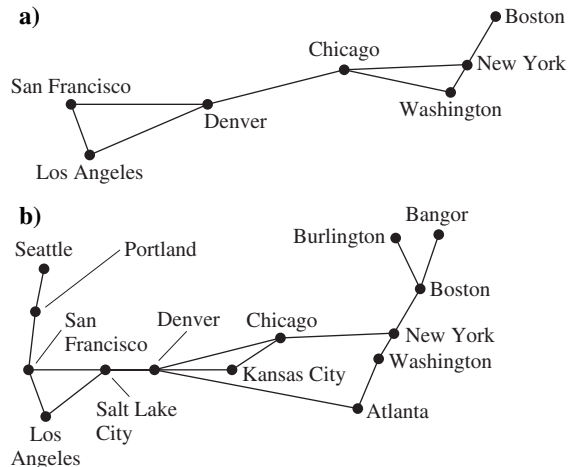
24. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.
25. Find the number of paths of length n between any two nonadjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.
26. Find the number of paths between c and d in the graph in Figure 1 of length
- a) 2. b) 3. c) 4. d) 5. e) 6. f) 7.
27. Find the number of paths from a to e in the directed graph in Exercise 2 of length
- a) 2. b) 3. c) 4. d) 5. e) 6. f) 7.
- *28. Show that every connected graph with n vertices has at least $n - 1$ edges.
29. Let $G = (V, E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that $u = v$. Show that R is an equivalence relation.
- *30. Show that in every simple graph there is a path from every vertex of odd degree to some other vertex of odd degree.

In Exercises 31–33 find all the cut vertices of the given graph.



34. Find all the cut edges in the graphs in Exercises 31–33.
- *35. Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.
- *36. Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v , both different from c , such that every path between u and v passes through c .
- *37. Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.
- *38. Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

39. A communications link in a network should be provided with a backup link if its failure makes it impossible for some message to be sent. For each of the communications networks shown here in (a) and (b), determine those links that should be backed up.



A **vertex basis** in a directed graph G is a minimal set B of vertices of G such that for each vertex v of G not in B there is a path to v from some vertex B .

40. Find a vertex basis for each of the directed graphs in Exercises 7–9 of Section 10.2.
41. What is the significance of a vertex basis in an influence graph (described in Example 2 of Section 10.1)? Find a vertex basis in the influence graph in that example.
42. Show that if a connected simple graph G is the union of the graphs G_1 and G_2 , then G_1 and G_2 have at least one common vertex.
- *43. Show that if a simple graph G has k connected components and these components have n_1, n_2, \dots, n_k vertices, respectively, then the number of edges of G does not exceed

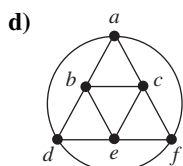
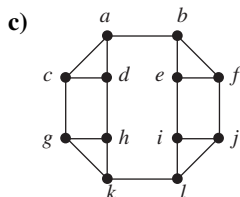
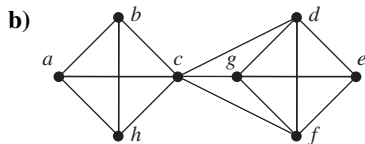
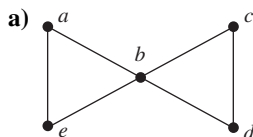
$$\sum_{i=1}^k C(n_i, 2).$$

- *44. Use Exercise 43 to show that a simple graph with n vertices and k connected components has at most $(n - k)(n - k + 1)/2$ edges. [Hint: First show that

$$\sum_{i=1}^k n_i^2 \leq n^2 - (k - 1)(2n - k),$$

where n_i is the number of vertices in the i th connected component.]

- *45. Show that a simple graph G with n vertices is connected if it has more than $(n - 1)(n - 2)/2$ edges.
46. Describe the adjacency matrix of a graph with n connected components when the vertices of the graph are listed so that vertices in each connected component are listed successively.
47. How many nonisomorphic connected simple graphs are there with n vertices when n is
 a) 2? b) 3? c) 4? d) 5?
48. Show that each of the following graphs has no cut vertices.
 a) C_n where $n \geq 3$
 b) W_n where $n \geq 3$
 c) $K_{m,n}$ where $m \geq 2$ and $n \geq 2$
 d) Q_n where $n \geq 2$
49. Show that each of the graphs in Exercise 48 has no cut edges.
50. For each of these graphs, find $\kappa(G)$, $\lambda(G)$, and $\min_{v \in V} \deg(v)$, and determine which of the two inequalities in $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$ are strict.



51. Show that if G is a connected graph, then it is possible to remove vertices to disconnect G if and only if G is not a complete graph.
52. Show that if G is a connected graph with n vertices then
 a) $\kappa(G) = n - 1$ if and only if $G = K_n$.
 b) $\lambda(G) = n - 1$ if and only if $G = K_n$.

53. Find $\kappa(K_{m,n})$ and $\lambda(K_{m,n})$, where m and n are positive integers.

54. Construct a graph G with $\kappa(G) = 1$, $\lambda(G) = 2$, and $\min_{v \in V} \deg(v) = 3$.

- *55. Show that if G is a graph, then $\kappa(G) \leq \lambda(G)$.

56. Explain how Theorem 2 can be used to find the length of the shortest path from a vertex v to a vertex w in a graph.

57. Use Theorem 2 to find the length of the shortest path between a and f in the graph in Figure 1.

58. Use Theorem 2 to find the length of the shortest path from a to c in the directed graph in Exercise 2.

59. Let P_1 and P_2 be two simple paths between the vertices u and v in the simple graph G that do not contain the same set of edges. Show that there is a simple circuit in G .

60. Show that the existence of a simple circuit of length k , where k is an integer greater than 2, is an invariant for graph isomorphism.

61. Explain how Theorem 2 can be used to determine whether a graph is connected.

62. Use Exercise 61 to show that the graph G_1 in Figure 2 is connected whereas the graph G_2 in that figure is not connected.

63. Show that a simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

64. In an old puzzle attributed to Alcuin of York (735–804), a farmer needs to carry a wolf, a goat, and a cabbage across a river. The farmer only has a small boat, which can carry the farmer and only one object (an animal or a vegetable). He can cross the river repeatedly. However, if the farmer is on the other shore, the wolf will eat the goat, and, similarly, the goat will eat the cabbage. We can describe each state by listing what is on each shore. For example, we can use the pair (FG, WC) for the state where the farmer and goat are on the first shore and the wolf and cabbage are on the other shore. [The symbol \emptyset is used when nothing is on a shore, so that $(FWGC, \emptyset)$ is the initial state.]

- a) Find all allowable states of the puzzle, where neither the wolf and the goat nor the goat and the cabbage are left on the same shore without the farmer.
- b) Construct a graph such that each vertex of this graph represents an allowable state and the vertices representing two allowable states are connected by an edge if it is possible to move from one state to the other using one trip of the boat.
- c) Explain why finding a path from the vertex representing $(FWGC, \emptyset)$ to the vertex representing $(\emptyset, FWGC)$ solves the puzzle.
- d) Find two different solutions of the puzzle, each using seven crossings.
- e) Suppose that the farmer must pay a toll of one dollar whenever he crosses the river with an animal. Which solution of the puzzle should the farmer use to pay the least total toll?

- *65. Use a graph model and a path in your graph, as in Exercise 64, to solve the **jealous husbands problem**. Two married couples, each a husband and a wife, want to cross a river. They can only use a boat that can carry one or two people from one shore to the other shore. Each husband is extremely jealous and is not willing to leave his wife with the other husband, either in the boat or on shore. How can these four people reach the opposite shore?
66. Suppose that you have a three-gallon jug and a five-gallon jug. You may fill either jug with water, you may empty either jug, and you may transfer water from either jug into the other jug. Use a path in a directed graph to show that you can end up with a jug containing exactly one gallon. [Hint: Use an ordered pair (a, b) to indicate how much water is in each jug. Represent these ordered pairs by vertices. Add an edge for each allowable operation with the jugs.]

10.5 Euler and Hamilton Paths

Introduction

Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once? Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once? Although these questions seem to be similar, the first question, which asks whether a graph has an *Euler circuit*, can be easily answered simply by examining the degrees of the vertices of the graph, while the second question, which asks whether a graph has a *Hamilton circuit*, is quite difficult to solve for most graphs. In this section we will study these questions and discuss the difficulty of solving them. Although both questions have many practical applications in many different areas, both arose in old puzzles. We will learn about these old puzzles as well as modern practical applications.

Euler Paths and Circuits

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure 1 depicts these regions and bridges.

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.

The Swiss mathematician Leonhard Euler solved this problem. His solution, published in 1736, may be the first use of graph theory. (For a translation of Euler's original paper see [BiLiWi99].) Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges. This multigraph is shown in Figure 2.



Only five bridges connect Kaliningrad today. Of these, just two remain from Euler's day.

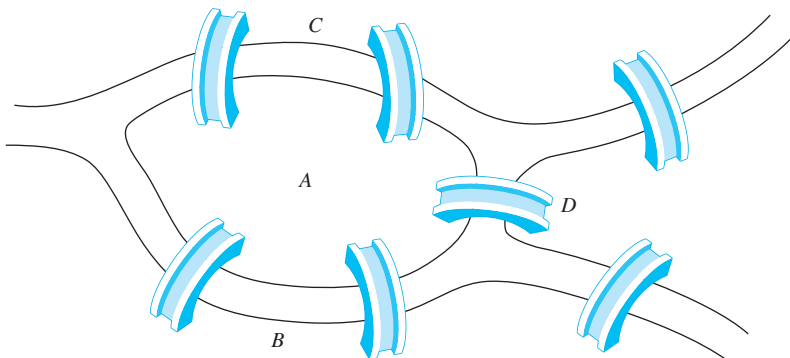


FIGURE 1 The Seven Bridges of Königsberg.

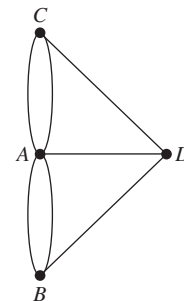


FIGURE 2 Multigraph Model of the Town of Königsberg.

complexity would be a major accomplishment because it has been shown that this problem is NP-complete (see Section 3.3). Consequently, the existence of such an algorithm would imply that many other seemingly intractable problems could be solved using algorithms with polynomial worst-case time complexity.

Applications of Hamilton Circuits

Hamilton paths and circuits can be used to solve practical problems. For example, many applications ask for a path or circuit that visits each road intersection in a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once. Finding a Hamilton path or circuit in the appropriate graph model can solve such problems. The famous **traveling salesperson problem** or **TSP** (also known in older literature as the **traveling salesman problem**) asks for the shortest route a traveling salesperson should take to visit a set of cities. This problem reduces to finding a Hamilton circuit in a complete graph such that the total weight of its edges is as small as possible. We will return to this question in Section 10.6.

We now describe a less obvious application of Hamilton circuits to coding.

EXAMPLE 8 Gray Codes The position of a rotating pointer can be represented in digital form. One way to do this is to split the circle into 2^n arcs of equal length and to assign a bit string of length n to each arc. Two ways to do this using bit strings of length three are shown in Figure 12.

The digital representation of the position of the pointer can be determined using a set of n contacts. Each contact is used to read one bit in the digital representation of the position. This is illustrated in Figure 13 for the two assignments from Figure 12.

When the pointer is near the boundary of two arcs, a mistake may be made in reading its position. This may result in a major error in the bit string read. For instance, in the coding scheme in Figure 12(a), if a small error is made in determining the position of the pointer, the bit string 100 is read instead of 011. All three bits are incorrect! To minimize the effect of an error in determining the position of the pointer, the assignment of the bit strings to the 2^n arcs should be made so that only one bit is different in the bit strings represented by adjacent arcs. This is exactly the situation in the coding scheme in Figure 12(b). An error in determining the position of the pointer gives the bit string 010 instead of 011. Only one bit is wrong.

A **Gray code** is a labeling of the arcs of the circle such that adjacent arcs are labeled with bit strings that differ in exactly one bit. The assignment in Figure 12(b) is a Gray code. We can find a Gray code by listing all bit strings of length n in such a way that each string differs in exactly one position from the preceding bit string, and the last string differs from the first in exactly one position. We can model this problem using the n -cube Q_n . What is needed to solve this problem is a Hamilton circuit in Q_n . Such Hamilton circuits are easily found. For instance, a Hamilton

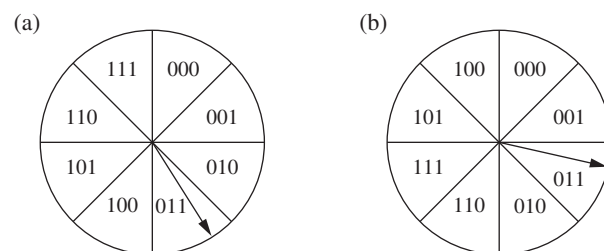


FIGURE 12 Converting the Position of a Pointer into Digital Form.

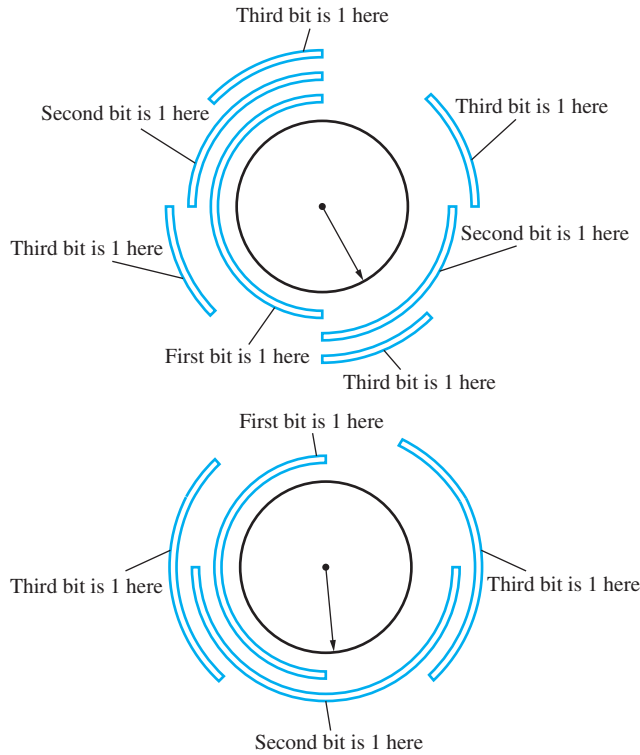


FIGURE 13 The Digital Representation of the Position of the Pointer.

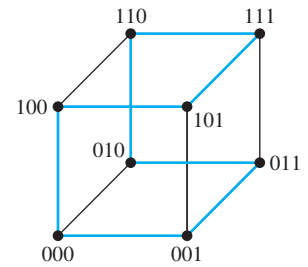


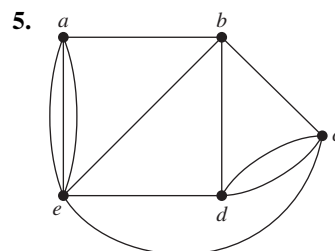
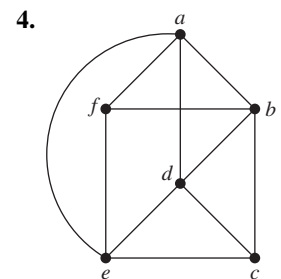
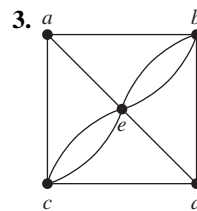
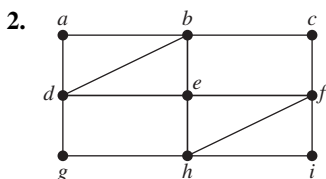
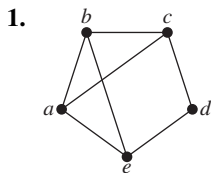
FIGURE 14 A Hamilton Circuit for Q_3 .

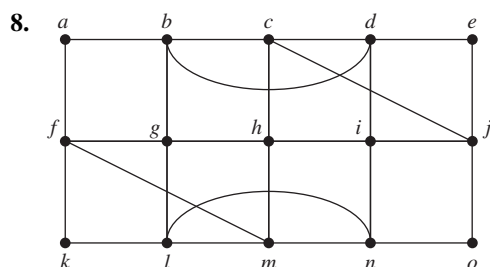
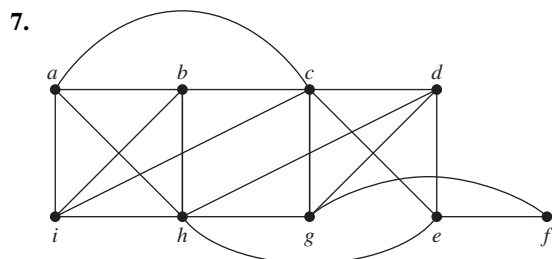
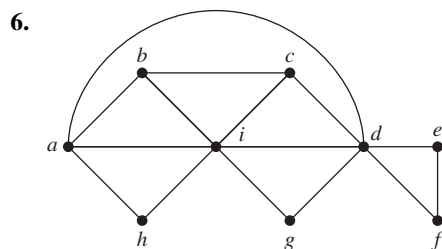
circuit for Q_3 is displayed in Figure 14. The sequence of bit strings differing in exactly one bit produced by this Hamilton circuit is 000, 001, 011, 010, 110, 111, 101, 100.

Gray codes are named after Frank Gray, who invented them in the 1940s at AT&T Bell Laboratories to minimize the effect of errors in transmitting digital signals. ◀

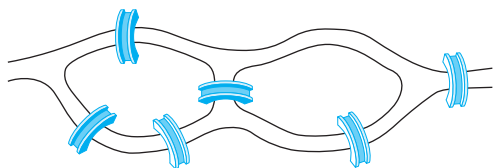
Exercises

In Exercises 1–8 determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



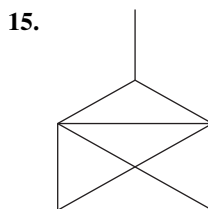
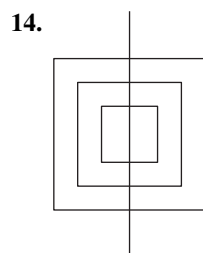
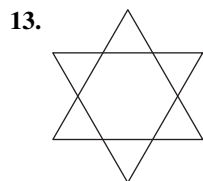


9. Suppose that in addition to the seven bridges of Königsberg (shown in Figure 1) there were two additional bridges, connecting regions B and C and regions B and D , respectively. Could someone cross all nine of these bridges exactly once and return to the starting point?
10. Can someone cross all the bridges shown in this map exactly once and return to the starting point?



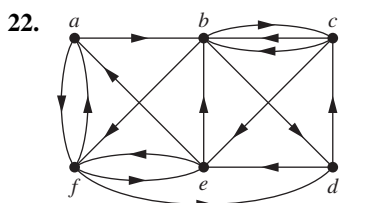
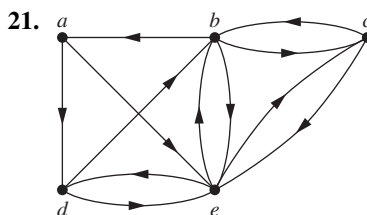
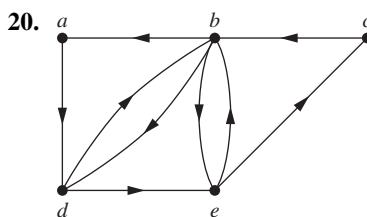
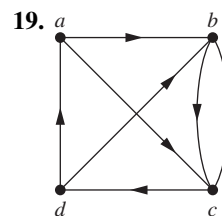
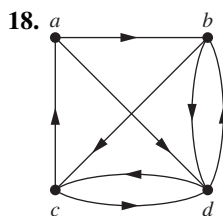
11. When can the centerlines of the streets in a city be painted without traveling a street more than once? (Assume that all the streets are two-way streets.)
12. Devise a procedure, similar to Algorithm 1, for constructing Euler paths in multigraphs.

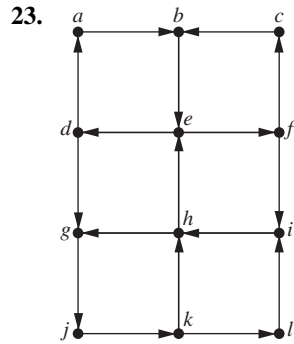
In Exercises 13–15 determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.



- *16.** Show that a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.
- *17.** Show that a directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree one larger than its out-degree and the other that has out-degree one larger than its in-degree.

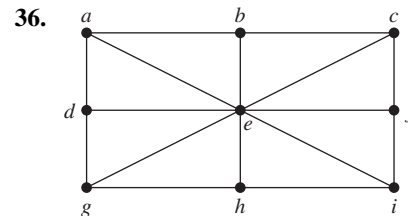
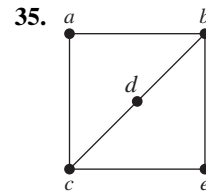
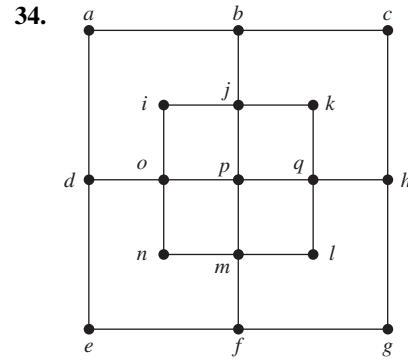
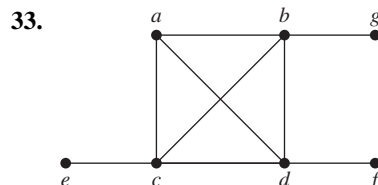
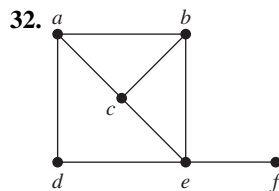
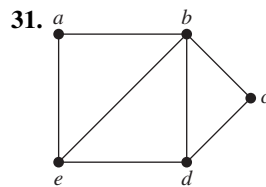
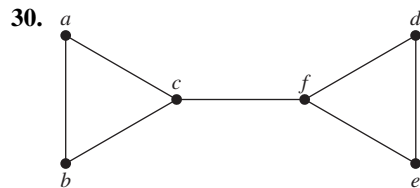
In Exercises 18–23 determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if one exists.





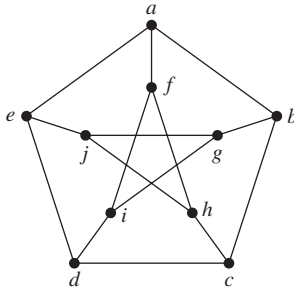
- *24. Devise an algorithm for constructing Euler circuits in directed graphs.
25. Devise an algorithm for constructing Euler paths in directed graphs.
26. For which values of n do these graphs have an Euler circuit?
 a) K_n b) C_n c) W_n d) Q_n
27. For which values of n do the graphs in Exercise 26 have an Euler path but no Euler circuit?
28. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an
 a) Euler circuit?
 b) Euler path?
29. Find the least number of times it is necessary to lift a pencil from the paper when drawing each of the graphs in Exercises 1–7 without retracing any part of the graph.

In Exercises 30–36 determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

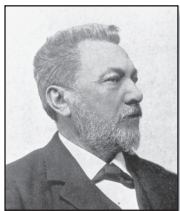
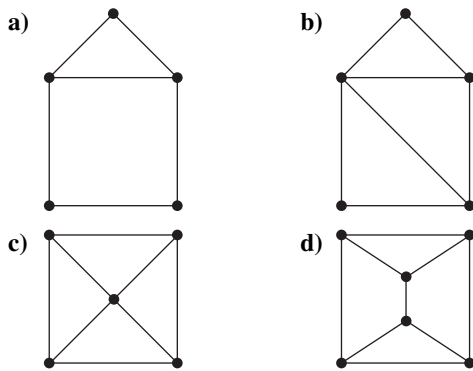


37. Does the graph in Exercise 30 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.
38. Does the graph in Exercise 31 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.
39. Does the graph in Exercise 32 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.
40. Does the graph in Exercise 33 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.
- *41. Does the graph in Exercise 34 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.
42. Does the graph in Exercise 35 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.
43. Does the graph in Exercise 36 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.
44. For which values of n do the graphs in Exercise 26 have a Hamilton circuit?
45. For which values of m and n does the complete bipartite graph $K_{m,n}$ have a Hamilton circuit?

- *46. Show that the **Petersen graph**, shown here, does not have a Hamilton circuit, but that the subgraph obtained by deleting a vertex v , and all edges incident with v , does have a Hamilton circuit.



47. For each of these graphs, determine (i) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit, (ii) whether Ore's theorem can be used to show that the graph has a Hamilton circuit, and (iii) whether the graph has a Hamilton circuit.



JULIUS PETER CHRISTIAN PETERSEN (1839–1910) Julius Petersen was born in the Danish town of Sorø. His father was a dyer. In 1854 his parents were no longer able to pay for his schooling, so he became an apprentice in an uncle's grocery store. When this uncle died, he left Petersen enough money to return to school. After graduating, he began studying engineering at the Polytechnical School in Copenhagen, later deciding to concentrate on mathematics. He published his first textbook, a book on logarithms, in 1858. When his inheritance ran out, he had to teach to make a living. From 1859 until 1871 Petersen taught at a prestigious private high school in Copenhagen. While teaching high school he continued his studies, entering Copenhagen University in 1862. He married Laura Bertelsen in 1862; they had three children, two sons and a daughter.

Petersen obtained a mathematics degree from Copenhagen University in 1866 and finally obtained his doctorate in 1871 from that school. After receiving his doctorate, he taught at a polytechnic and military academy. In 1887 he was appointed to a professorship at the University of Copenhagen. Petersen was well known in Denmark as the author of a large series of textbooks for high schools and universities. One of his books, *Methods and Theories for the Solution of Problems of Geometrical Construction*, was translated into eight languages, with the English language version last reprinted in 1960 and the French version reprinted as recently as 1990, more than a century after the original publication date.

Petersen worked in a wide range of areas, including algebra, analysis, cryptography, geometry, mechanics, mathematical economics, and number theory. His contributions to graph theory, including results on regular graphs, are his best-known work. He was noted for his clarity of exposition, problem-solving skills, originality, sense of humor, vigor, and teaching. One interesting fact about Petersen was that he preferred not to read the writings of other mathematicians. This led him often to rediscover results already proved by others, often with embarrassing consequences. However, he was often angry when other mathematicians did not read his writings!

Petersen's death was front-page news in Copenhagen. A newspaper of the time described him as the Hans Christian Andersen of science—a child of the people who made good in the academic world.

48. Can you find a simple graph with n vertices with $n \geq 3$ that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n - 1)/2$?

- *49. Show that there is a Gray code of order n whenever n is a positive integer, or equivalently, show that the n -cube Q_n , $n > 1$, always has a Hamilton circuit. [Hint: Use mathematical induction. Show how to produce a Gray code of order n from one of order $n - 1$.]



Fleury's algorithm, published in 1883, constructs Euler circuits by first choosing an arbitrary vertex of a connected multigraph, and then forming a circuit by choosing edges successively. Once an edge is chosen, it is removed. Edges are chosen successively so that each edge begins where the last edge ends, and so that this edge is not a cut edge unless there is no alternative.

50. Use Fleury's algorithm to find an Euler circuit in the graph G in Figure 5.

- *51. Express Fleury's algorithm in pseudocode.

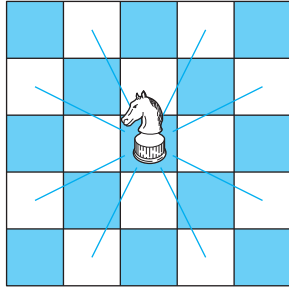
- **52. Prove that Fleury's algorithm always produces an Euler circuit.

- *53. Give a variant of Fleury's algorithm to produce Euler paths.

54. A diagnostic message can be sent out over a computer network to perform tests over all links and in all devices. What sort of paths should be used to test all links? To test all devices?

55. Show that a bipartite graph with an odd number of vertices does not have a Hamilton circuit.

A **knight** is a chess piece that can move either two spaces horizontally and one space vertically or one space horizontally and two spaces vertically. That is, a knight on square (x, y) can move to any of the eight squares $(x \pm 2, y \pm 1)$, $(x \pm 1, y \pm 2)$, if these squares are on the chessboard, as illustrated here.



A **knight's tour** is a sequence of legal moves by a knight starting at some square and visiting each square exactly once. A knight's tour is called **reentrant** if there is a legal move that takes the knight from the last square of the tour back to where the tour began. We can model knight's tours using the graph that has a vertex for each square on the board, with an edge connecting two vertices if a knight can legally move between the squares represented by these vertices.

56. Draw the graph that represents the legal moves of a knight on a 3×3 chessboard.
57. Draw the graph that represents the legal moves of a knight on a 3×4 chessboard.
58. a) Show that finding a knight's tour on an $m \times n$ chessboard is equivalent to finding a Hamilton path on the graph representing the legal moves of a knight on that board.
b) Show that finding a reentrant knight's tour on an $m \times n$ chessboard is equivalent to finding a Hamilton circuit on the corresponding graph.
- *59. Show that there is a knight's tour on a 3×4 chessboard.
- *60. Show that there is no knight's tour on a 3×3 chessboard.
- *61. Show that there is no knight's tour on a 4×4 chessboard.
62. Show that the graph representing the legal moves of a knight on an $m \times n$ chessboard, whenever m and n are positive integers, is bipartite.
63. Show that there is no reentrant knight's tour on an $m \times n$ chessboard when m and n are both odd. [Hint: Use Exercises 55, 58b, and 62.]
- *64. Show that there is a knight's tour on an 8×8 chessboard. [Hint: You can construct a knight's tour using a method invented by H. C. Warnsdorff in 1823: Start in any square, and then always move to a square connected to the fewest number of unused squares. Although this method may not always produce a knight's tour, it often does.]
65. The parts of this exercise outline a proof of Ore's theorem. Suppose that G is a simple graph with n vertices, $n \geq 3$, and $\deg(x) + \deg(y) \geq n$ whenever x and y are nonadjacent vertices in G . Ore's theorem states that under these conditions, G has a Hamilton circuit.
 - a) Show that if G does not have a Hamilton circuit, then there exists another graph H with the same vertices as G , which can be constructed by adding edges to G such that the addition of a single edge would produce a Hamilton circuit in H . [Hint: Add as many edges as possible at each successive vertex of G without producing a Hamilton circuit.]
 - b) Show that there is a Hamilton path in H .
 - c) Let v_1, v_2, \dots, v_n be a Hamilton path in H . Show that $\deg(v_1) + \deg(v_n) \geq n$ and that there are at most $\deg(v_1)$ vertices not adjacent to v_n (including v_n itself).
 - d) Let S be the set of vertices preceding each vertex adjacent to v_1 in the Hamilton path. Show that S contains $\deg(v_1)$ vertices and $v_n \notin S$.
 - e) Show that S contains a vertex v_k , which is adjacent to v_n , implying that there are edges connecting v_1 and v_{k+1} and v_k and v_n .
 - f) Show that part (e) implies that $v_1, v_2, \dots, v_{k-1}, v_k, v_n, v_{n-1}, \dots, v_{k+1}, v_1$ is a Hamilton circuit in G . Conclude from this contradiction that Ore's theorem holds.
- *66. Show that the worst case computational complexity of Algorithm 1 for finding Euler circuits in a connected graph with all vertices of even degree is $O(m)$, where m is the number of edges of G .

10.6 Shortest-Path Problems

Introduction

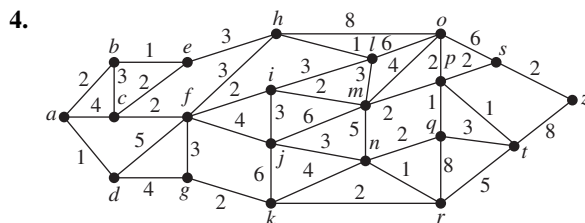
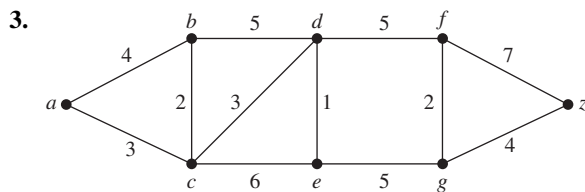
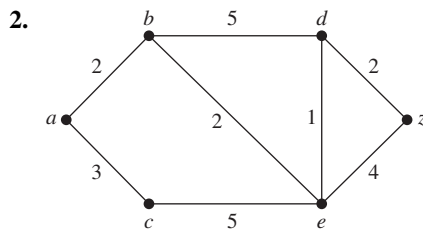
Many problems can be modeled using graphs with weights assigned to their edges. As an illustration, consider how an airline system can be modeled. We set up the basic graph model by representing cities by vertices and flights by edges. Problems involving distances can be modeled by assigning distances between cities to the edges. Problems involving flight time can be modeled by assigning flight times to edges. Problems involving fares can be modeled by

In practice, algorithms have been developed that can solve traveling salesperson problems with as many as 1000 vertices within 2% of an exact solution using only a few minutes of computer time. For more information about the traveling salesperson problem, including history, applications, and algorithms, see the chapter on this topic in *Applications of Discrete Mathematics* [MiRo91] also available on the website for this book.

Exercises

- For each of these problems about a subway system, describe a weighted graph model that can be used to solve the problem.
 - What is the least amount of time required to travel between two stops?
 - What is the minimum distance that can be traveled to reach a stop from another stop?
 - What is the least fare required to travel between two stops if fares between stops are added to give the total fare?

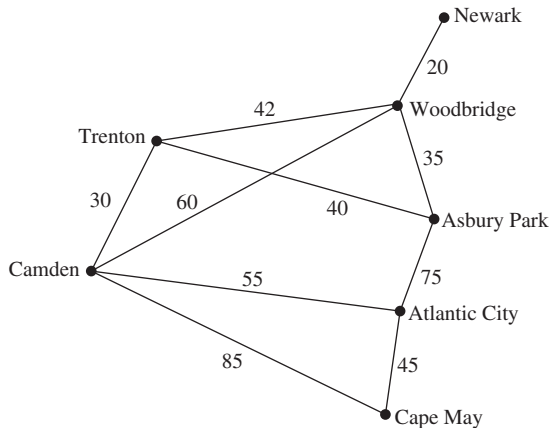
In Exercises 2–4 find the length of a shortest path between a and z in the given weighted graph.



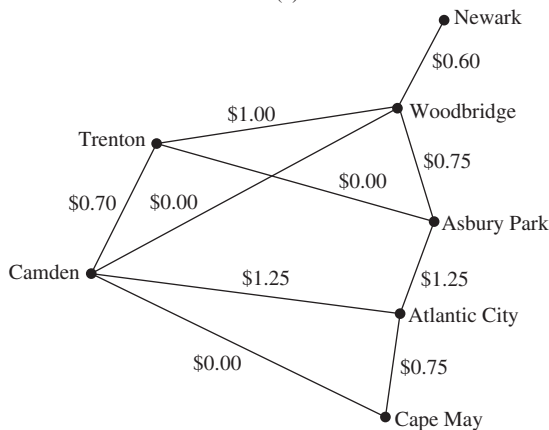
- Find a shortest path between a and z in each of the weighted graphs in Exercises 2–4.
- Find the length of a shortest path between these pairs of vertices in the weighted graph in Exercise 3.
 - a and d
 - a and f
 - c and f
 - b and z

- Find shortest paths in the weighted graph in Exercise 3 between the pairs of vertices in Exercise 6.
- Find a shortest path (in mileage) between each of the following pairs of cities in the airline system shown in Figure 1.
 - New York and Los Angeles
 - Boston and San Francisco
 - Miami and Denver
 - Miami and Los Angeles
- Find a combination of flights with the least total air time between the pairs of cities in Exercise 8, using the flight times shown in Figure 1.
- Find a least expensive combination of flights connecting the pairs of cities in Exercise 8, using the fares shown in Figure 1.
- Find a shortest route (in distance) between computer centers in each of these pairs of cities in the communications network shown in Figure 2.
 - Boston and Los Angeles
 - New York and San Francisco
 - Dallas and San Francisco
 - Denver and New York
- Find a route with the shortest response time between the pairs of computer centers in Exercise 11 using the response times given in Figure 2.
- Find a least expensive route, in monthly lease charges, between the pairs of computer centers in Exercise 11 using the lease charges given in Figure 2.
- Explain how to find a path with the least number of edges between two vertices in an undirected graph by considering it as a shortest path problem in a weighted graph.
- Extend Dijkstra's algorithm for finding the length of a shortest path between two vertices in a weighted simple connected graph so that the length of a shortest path between the vertex a and every other vertex of the graph is found.
- Extend Dijkstra's algorithm for finding the length of a shortest path between two vertices in a weighted simple connected graph so that a shortest path between these vertices is constructed.

17. The weighted graphs in the figures here show some major roads in New Jersey. Part (a) shows the distances between cities on these roads; part (b) shows the tolls.



(a)



(b)

- a) Find a shortest route in distance between Newark and Camden, and between Newark and Cape May, using these roads.
- b) Find a least expensive route in terms of total tolls using the roads in the graph between the pairs of cities in part (a) of this exercise.
18. Is a shortest path between two vertices in a weighted graph unique if the weights of edges are distinct?
19. What are some applications where it is necessary to find the length of a longest simple path between two vertices in a weighted graph?
20. What is the length of a longest simple path in the weighted graph in Figure 4 between a and z ? Between c and z ?



Floyd's algorithm, displayed as Algorithm 2, can be used to find the length of a shortest path between all pairs of vertices in a weighted connected simple graph. However, this algorithm cannot be used to construct shortest paths. (We assign an infinite weight to any pair of vertices not connected by an edge in the graph.)

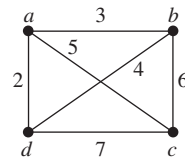
21. Use Floyd's algorithm to find the distance between all pairs of vertices in the weighted graph in Figure 4(a).
- *22. Prove that Floyd's algorithm determines the shortest distance between all pairs of vertices in a weighted simple graph.
- *23. Give a big- O estimate of the number of operations (comparisons and additions) used by Floyd's algorithm to determine the shortest distance between every pair of vertices in a weighted simple graph with n vertices.
- *24. Show that Dijkstra's algorithm may not work if edges can have negative weights.

ALGORITHM 2 Floyd's Algorithm.

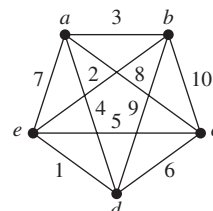
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procedure Floyd( $G$ : weighted simple graph)
  {  $G$  has vertices  $v_1, v_2, \dots, v_n$  and weights  $w(v_i, v_j)$ 
    with  $w(v_i, v_j) = \infty$  if  $\{v_i, v_j\}$  is not an edge }
  for  $i := 1$  to  $n$ 
    for  $j := 1$  to  $n$ 
       $d(v_i, v_j) := w(v_i, v_j)$ 
  for  $i := 1$  to  $n$ 
    for  $j := 1$  to  $n$ 
      for  $k := 1$  to  $n$ 
        if  $d(v_j, v_i) + d(v_i, v_k) < d(v_j, v_k)$ 
          then  $d(v_j, v_k) := d(v_j, v_i) + d(v_i, v_k)$ 
  return [ $d(v_i, v_j)$ ] {  $d(v_i, v_j)$  is the length of a shortest
    path between  $v_i$  and  $v_j$  for  $1 \leq i \leq n, 1 \leq j \leq n$  }
  
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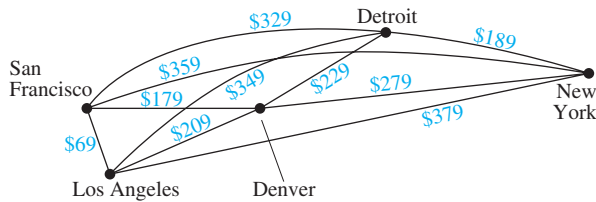
25. Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



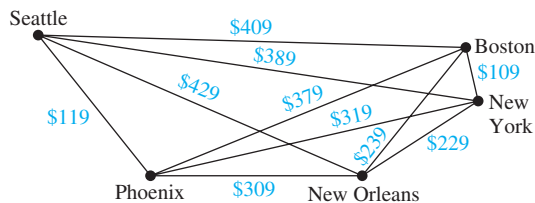
26. Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



27. Find a route with the least total airfare that visits each of the cities in this graph, where the weight on an edge is the least price available for a flight between the two cities.



28. Find a route with the least total airfare that visits each of the cities in this graph, where the weight on an edge is the least price available for a flight between the two cities.



29. Construct a weighted undirected graph such that the total weight of a circuit that visits every vertex at least once is minimized for a circuit that visits some vertices more than once. [Hint: There are examples with three vertices.]

30. Show that the problem of finding a circuit of minimum total weight that visits every vertex of a weighted graph at least once can be reduced to the problem of finding a circuit of minimum total weight that visits each vertex of a weighted graph exactly once. Do so by constructing a new weighted graph with the same vertices and edges as the original graph but whose weight of the edge connecting the vertices u and v is equal to the minimum total weight of a path from u to v in the original graph.

- *31. The **longest path problem** in a weighted directed graph with no simple circuits asks for a path in this graph such that the sum of its edge weights is a maximum. Devise an algorithm for solving the longest path problem. [Hint: First find a topological ordering of the vertices of the graph.]

10.7 Planar Graphs

Introduction



Consider the problem of joining three houses to each of three separate utilities, as shown in Figure 1. Is it possible to join these houses and utilities so that none of the connections cross? This problem can be modeled using the complete bipartite graph $K_{3,3}$. The original question can be rephrased as: Can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?

In this section we will study the question of whether a graph can be drawn in the plane without edges crossing. In particular, we will answer the houses-and-utilities problem.

There are always many ways to represent a graph. When is it possible to find at least one way to represent this graph in a plane without any edges crossing?

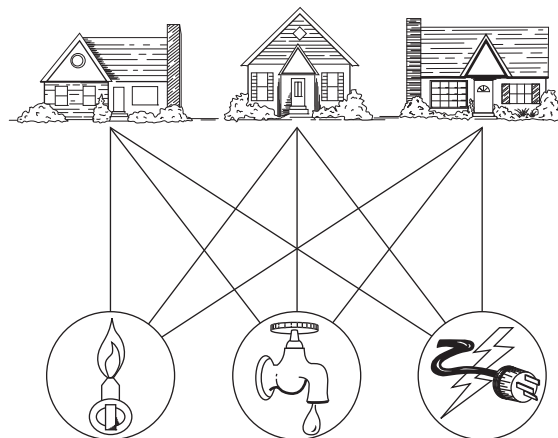


FIGURE 1 Three Houses and Three Utilities.

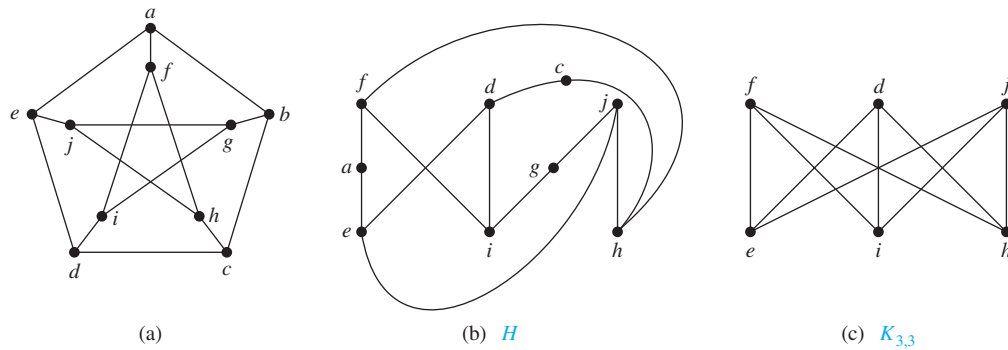


FIGURE 14 (a) The Petersen Graph, (b) a Subgraph H Homeomorphic to $K_{3,3}$, and (c) $K_{3,3}$.

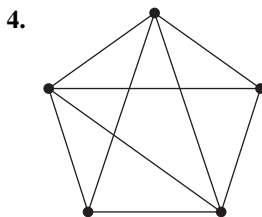
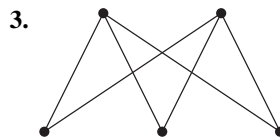
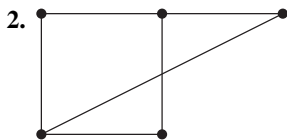
EXAMPLE 9 Is the Petersen graph, shown in Figure 14(a), planar? (The Danish mathematician Julius Petersen studied this graph in 1891; it is often used to illustrate various theoretical properties of graphs.)

Solution: The subgraph H of the Petersen graph obtained by deleting b and the three edges that have b as an endpoint, shown in Figure 14(b), is homeomorphic to $K_{3,3}$, with vertex sets $\{f, d, j\}$ and $\{e, i, h\}$, because it can be obtained by a sequence of elementary subdivisions, deleting $\{d, h\}$ and adding $\{c, h\}$ and $\{c, d\}$, deleting $\{e, f\}$ and adding $\{a, e\}$ and $\{a, f\}$, and deleting $\{i, j\}$ and adding $\{g, i\}$ and $\{g, j\}$. Hence, the Petersen graph is not planar. ▶

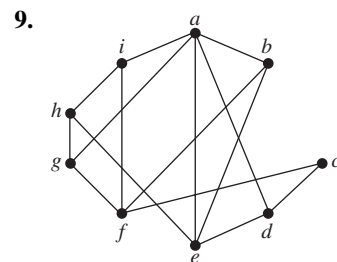
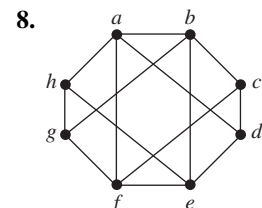
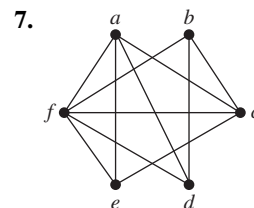
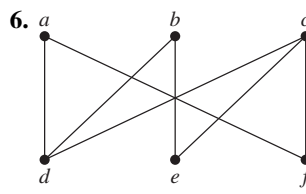
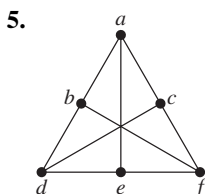
Exercises

1. Can five houses be connected to two utilities without connections crossing?

In Exercises 2–4 draw the given planar graph without any crossings.



In Exercises 5–9 determine whether the given graph is planar. If so, draw it so that no edges cross.



10. Complete the argument in Example 3.

11. Show that K_5 is nonplanar using an argument similar to that given in Example 3.

12. Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?

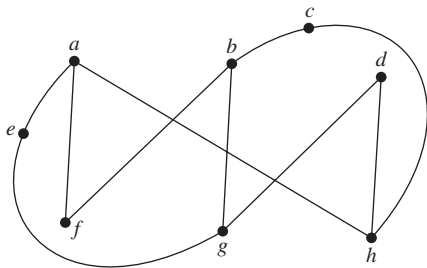
13. Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

14. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

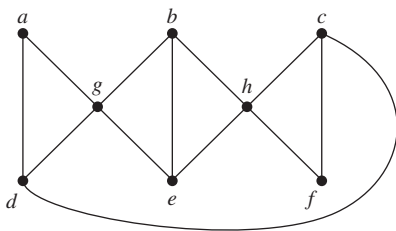
15. Prove Corollary 3.
16. Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v - 4$ if $v \geq 3$.
- *17. Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.
18. Suppose that a planar graph has k connected components, e edges, and v vertices. Also suppose that the plane is divided into r regions by a planar representation of the graph. Find a formula for r in terms of e , v , and k .
19. Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?
 a) K_5 b) K_6 c) $K_{3,3}$ d) $K_{3,4}$

In Exercises 20–22 determine whether the given graph is homeomorphic to $K_{3,3}$.

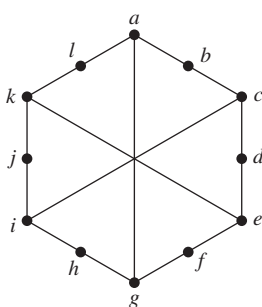
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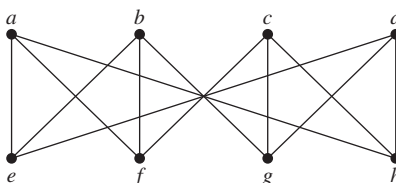


22.

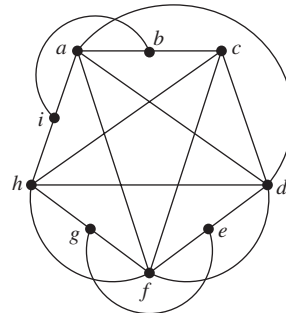


In Exercises 23–25 use Kuratowski's theorem to determine whether the given graph is planar.

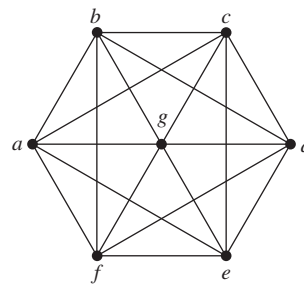
23.



24.



25.



The **crossing number** of a simple graph is the minimum number of crossings that can occur when this graph is drawn in the plane where no three arcs representing edges are permitted to cross at the same point.

26. Show that $K_{3,3}$ has 1 as its crossing number.

*27. Find the crossing numbers of each of these nonplanar graphs.

- a) K_5 b) K_6 c) K_7
 d) $K_{3,4}$ e) $K_{4,4}$ f) $K_{5,5}$

*28. Find the crossing number of the Petersen graph.

*29. Show that if m and n are even positive integers, the crossing number of $K_{m,n}$ is less than or equal to $mn(m-2)(n-2)/16$. [Hint: Place m vertices along the x -axis so that they are equally spaced and symmetric about the origin and place n vertices along the y -axis so that they are equally spaced and symmetric about the origin. Now connect each of the m vertices on the x -axis to each of the vertices on the y -axis and count the crossings.]

The **thickness** of a simple graph G is the smallest number of planar subgraphs of G that have G as their union.

30. Show that $K_{3,3}$ has 2 as its thickness.

*31. Find the thickness of the graphs in Exercise 27.

32. Show that if G is a connected simple graph with v vertices and e edges, where $v \geq 3$, then the thickness of G is at least $\lceil e/(3v-6) \rceil$.

*33. Use Exercise 32 to show that the thickness of K_n is at least $\lfloor (n+7)/6 \rfloor$ whenever n is a positive integer.

34. Show that if G is a connected simple graph with v vertices and e edges, where $v \geq 3$, and no circuits of length three, then the thickness of G is at least $\lceil e/(2v-4) \rceil$.

35. Use Exercise 34 to show that the thickness of $K_{m,n}$, where m and n are not both 1, is at least $\lceil mn/(2m+2n-4) \rceil$ whenever m and n are positive integers.

*36. Draw K_5 on the surface of a torus (a doughnut-shaped solid) so that no edges cross.

*37. Draw $K_{3,3}$ on the surface of a torus so that no edges cross.

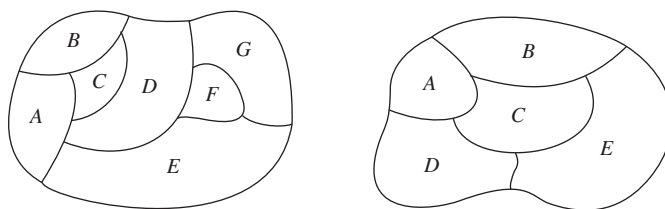


FIGURE 1 Two Maps.

10.8 Graph Coloring

Introduction



Problems related to the coloring of maps of regions, such as maps of parts of the world, have generated many results in graph theory. When a map* is colored, two regions with a common border are customarily assigned different colors. One way to ensure that two adjacent regions never have the same color is to use a different color for each region. However, this is inefficient, and on maps with many regions it would be hard to distinguish similar colors. Instead, a small number of colors should be used whenever possible. Consider the problem of determining the least number of colors that can be used to color a map so that adjacent regions never have the same color. For instance, for the map shown on the left in Figure 1, four colors suffice, but three colors are not enough. (The reader should check this.) In the map on the right in Figure 1, three colors are sufficient (but two are not).

Each map in the plane can be represented by a graph. To set up this correspondence, each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one point are not considered adjacent. The resulting graph is called the **dual graph** of the map. By the way in which dual graphs of maps are constructed, it is clear that any map in the plane has a planar dual graph. Figure 2 displays the dual graphs that correspond to the maps shown in Figure 1.

The problem of coloring the regions of a map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color. We now define a graph coloring.

DEFINITION 1

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

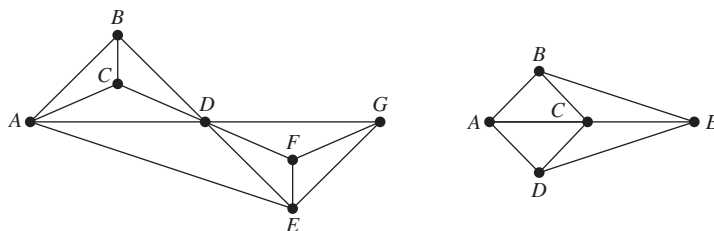


FIGURE 2 Dual Graphs of the Maps in Figure 1.

*We will assume that all regions in a map are connected. This eliminates any problems presented by such geographical entities as Michigan.

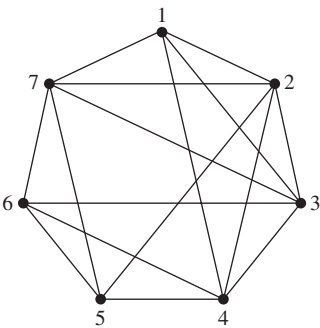


FIGURE 8 The Graph Representing the Scheduling of Final Exams.

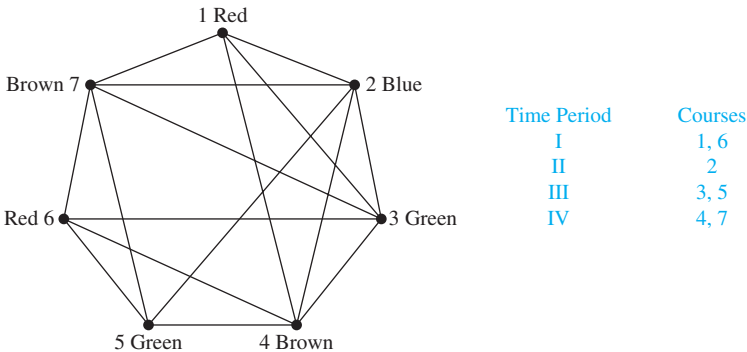


FIGURE 9 Using a Coloring to Schedule Final Exams.

Now consider an application to the assignment of television channels.

EXAMPLE 6 Frequency Assignments Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Solution: Construct a graph by assigning a vertex to each station. Two vertices are connected by an edge if they are located within 150 miles of each other. An assignment of channels corresponds to a coloring of the graph, where each color represents a different channel. ◀

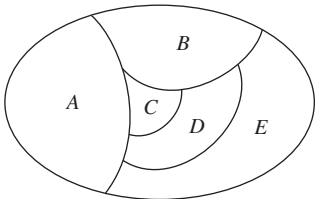
An application of graph coloring to compilers is considered in Example 7.

EXAMPLE 7 Index Registers In efficient compilers the execution of loops is speeded up when frequently used variables are stored temporarily in index registers in the central processing unit, instead of in regular memory. For a given loop, how many index registers are needed? This problem can be addressed using a graph coloring model. To set up the model, let each vertex of a graph represent a variable in the loop. There is an edge between two vertices if the variables they represent must be stored in index registers at the same time during the execution of the loop. Thus, the chromatic number of the graph gives the number of index registers needed, because different registers must be assigned to variables when the vertices representing these variables are adjacent in the graph. ◀

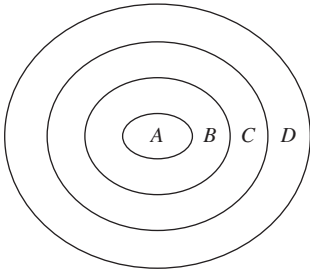
Exercises

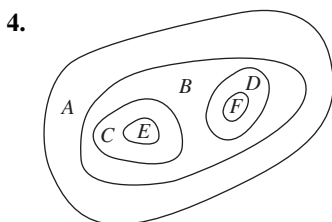
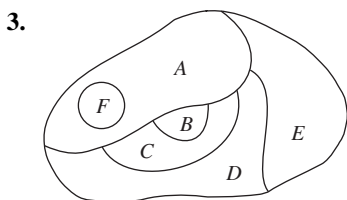
In Exercises 1–4 construct the dual graph for the map shown. Then find the number of colors needed to color the map so that no two adjacent regions have the same color.

1.

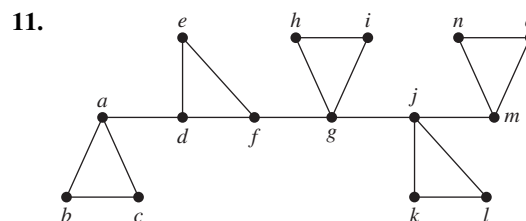
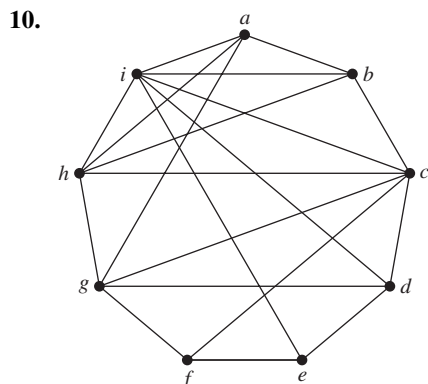
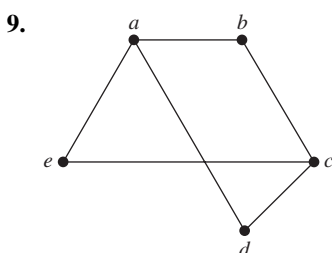
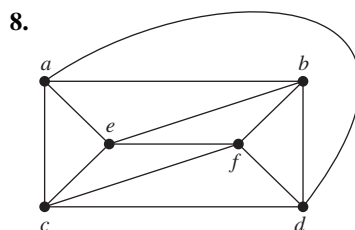
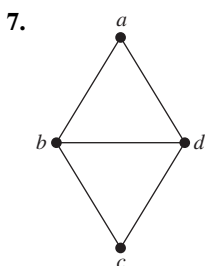
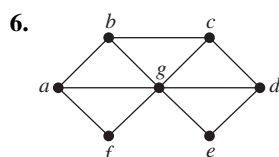
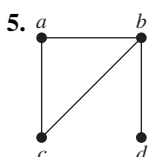


2.





In Exercises 5–11 find the chromatic number of the given graph.



12. For the graphs in Exercises 5–11, decide whether it is possible to decrease the chromatic number by removing a single vertex and all edges incident with it.

13. Which graphs have a chromatic number of 1?

14. What is the least number of colors needed to color a map of the United States? Do not consider adjacent states that meet only at a corner. Suppose that Michigan is one region. Consider the vertices representing Alaska and Hawaii as isolated vertices.

15. What is the chromatic number of W_n ?

16. Show that a simple graph that has a circuit with an odd number of vertices in it cannot be colored using two colors.

17. Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math 115 and CS 473, both Math 116 and CS 473, both Math 195 and CS 101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other pair of courses.

18. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

	1	2	3	4	5	6
1	—	85	175	200	50	100
2	85	—	125	175	100	160
3	175	125	—	100	200	250
4	200	175	100	—	210	220
5	50	100	200	210	—	100
6	100	160	250	220	100	—

19. The mathematics department has six committees, each meeting once a month. How many different meeting times must be used to ensure that no member is scheduled to attend two meetings at the same time if the committees are $C_1 = \{\text{Arlinghaus, Brand, Zaslavsky}\}$, $C_2 = \{\text{Brand, Lee, Rosen}\}$, $C_3 = \{\text{Arlinghaus, Rosen, Zaslavsky}\}$, $C_4 = \{\text{Lee, Rosen, Zaslavsky}\}$, $C_5 = \{\text{Arlinghaus, Brand}\}$, and $C_6 = \{\text{Brand, Rosen, Zaslavsky}\}$?

20. A zoo wants to set up natural habitats in which to exhibit its animals. Unfortunately, some animals will eat some of the others when given the opportunity. How can a graph model and a coloring be used to determine the number of different habitats needed and the placement of the animals in these habitats?

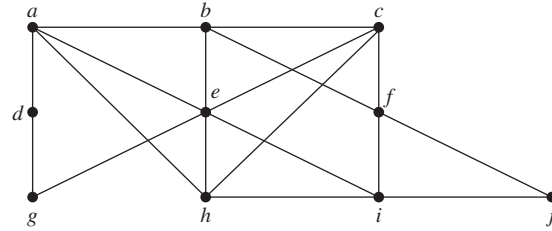


An **edge coloring** of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors. The **edge chromatic number** of a graph is the smallest number of colors that can be used in an edge coloring of the graph. The edge chromatic number of a graph G is denoted by $\chi'(G)$.

21. Find the edge chromatic number of each of the graphs in Exercises 5–11.
22. Suppose that n devices are on a circuit board and that these devices are connected by colored wires. Express the number of colors needed for the wires, in terms of the edge chromatic number of the graph representing this circuit board, under the requirement that the wires leaving a particular device must be different colors. Explain your answer.
23. Find the edge chromatic numbers of
- C_n , where $n \geq 3$.
 - W_n , where $n \geq 3$.
24. Show that the edge chromatic number of a graph must be at least as large as the maximum degree of a vertex of the graph.
25. Show that if G is a graph with n vertices, then no more than $n/2$ edges can be colored the same in an edge coloring of G .
- *26. Find the edge chromatic number of K_n when n is a positive integer.
27. Seven variables occur in a loop of a computer program. The variables and the steps during which they must be stored are t : steps 1 through 6; u : step 2; v : steps 2 through 4; w : steps 1, 3, and 5; x : steps 1 and 6; y : steps 3 through 6; and z : steps 4 and 5. How many different index registers are needed to store these variables during execution?
28. What can be said about the chromatic number of a graph that has K_n as a subgraph?

This algorithm can be used to color a simple graph: First, list the vertices $v_1, v_2, v_3, \dots, v_n$ in order of decreasing degree so that $\deg(v_1) \geq \deg(v_2) \geq \dots \geq \deg(v_n)$. Assign color 1 to v_1 and to the next vertex in the list not adjacent to v_1 (if one exists), and successively to each vertex in the list not adjacent to a vertex already assigned color 1. Then assign color 2 to the first vertex in the list not already colored. Successively assign color 2 to vertices in the list that have not already been colored and are not adjacent to vertices assigned color 2. If uncolored vertices remain, assign color 3 to the first vertex in the list not yet colored, and use color 3 to successively color those vertices not already colored and not adjacent to vertices assigned color 3. Continue this process until all vertices are colored.

29. Construct a coloring of the graph shown using this algorithm.

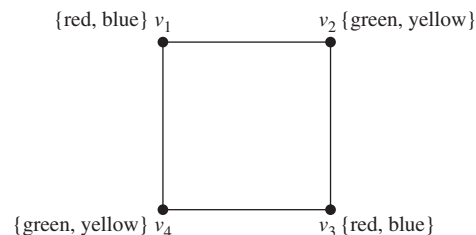


- *30. Use pseudocode to describe this coloring algorithm.
- *31. Show that the coloring produced by this algorithm may use more colors than are necessary to color a graph.

A connected graph G is called **chromatically k -critical** if the chromatic number of G is k , but for every edge of G , the chromatic number of the graph obtained by deleting this edge from G is $k - 1$.

32. Show that C_n is chromatically 3-critical whenever n is an odd positive integer, $n \geq 3$.
33. Show that W_n is chromatically 4-critical whenever n is an odd integer, $n \geq 3$.
34. Show that W_4 is not chromatically 3-critical.
35. Show that if G is a chromatically k -critical graph, then the degree of every vertex of G is at least $k - 1$.

A **k -tuple coloring** of a graph G is an assignment of a set of k different colors to each of the vertices of G such that no two adjacent vertices are assigned a common color. We denote by $\chi_k(G)$ the smallest positive integer n such that G has a k -tuple coloring using n colors. For example, $\chi_2(C_4) = 4$. To see this, note that using only four colors we can assign two colors to each vertex of C_4 , as illustrated, so that no two adjacent vertices are assigned the same color. Furthermore, no fewer than four colors suffice because the vertices v_1 and v_2 each must be assigned two colors, and a common color cannot be assigned to both v_1 and v_2 . (For more information about k -tuple coloring, see [MiRo91].)



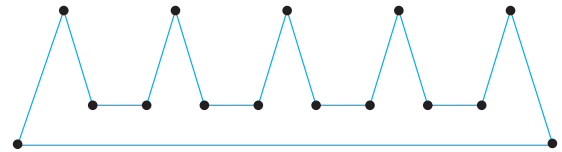
36. Find these values:
- | | | |
|-------------------|----------------------|------------------|
| a) $\chi_2(K_3)$ | b) $\chi_2(K_4)$ | c) $\chi_2(W_4)$ |
| d) $\chi_2(C_5)$ | e) $\chi_2(K_{3,4})$ | f) $\chi_3(K_5)$ |
| *g) $\chi_3(C_5)$ | h) $\chi_3(K_{4,5})$ | |

- *37. Let G and H be the graphs displayed in Figure 3. Find
- $\chi_2(G)$.
 - $\chi_2(H)$.
 - $\chi_3(G)$.
 - $\chi_3(H)$.
38. What is $\chi_k(G)$ if G is a bipartite graph and k is a positive integer?
39. Frequencies for mobile radio (or cellular) telephones are assigned by zones. Each zone is assigned a set of frequencies to be used by vehicles in that zone. The same frequency cannot be used in different zones when interference can occur between telephones in these zones. Explain how a k -tuple coloring can be used to assign k frequencies to each mobile radio zone in a region.
- *40. Show that every planar graph G can be colored using six or fewer colors. [Hint: Use mathematical induction on the number of vertices of the graph. Apply Corollary 2 of Section 10.7 to find a vertex v with $\deg(v) \leq 5$. Consider the subgraph of G obtained by deleting v and all edges incident with it.]
- **41. Show that every planar graph G can be colored using five or fewer colors. [Hint: Use the hint provided for Exercise 40.]

The famous Art Gallery Problem asks how many guards are needed to see all parts of an art gallery, where the gallery is the interior and boundary of a polygon with n sides. To state this problem more precisely, we need some terminology. A point x inside or on the boundary of a simple polygon P **covers** or **sees** a point y inside or on P if all points on the line segment xy are in the interior or on the boundary of P . We say that a set of points is a **guarding set** of a simple polygon P if for every point y inside P or on the boundary of P there is a point x in this guarding set that sees y . Denote by $G(P)$ the minimum number of points needed to guard the simple polygon P . The **art gallery problem** asks for the function $g(n)$, which is the maximum value of $G(P)$ over all simple polygons with n vertices. That is, $g(n)$ is the minimum positive integer for which

it is guaranteed that a simple polygon with n vertices can be guarded with $g(n)$ or fewer guards.

42. Show that $g(3) = 1$ and $g(4) = 1$ by showing that all triangles and quadrilaterals can be guarded using one point.
- *43. Show that $g(5) = 1$. That is, show that all pentagons can be guarded using one point. [Hint: Show that there are either 0, 1, or 2 vertices with an interior angle greater than 180 degrees and that in each case, one guard suffices.]
- *44. Show that $g(6) = 2$ by first using Exercises 42 and 43 as well as Lemma 1 in Section 5.2 to show that $g(6) \leq 2$ and then find a simple hexagon for which two guards are needed.
- *45. Show that $g(n) \geq \lfloor n/3 \rfloor$. [Hint: Consider the polygon with $3k$ vertices that resembles a comb with k prongs, such as the polygon with 15 sides shown here.]



- *46. Solve the art gallery problem by proving the **art gallery theorem**, which states that at most $\lfloor n/3 \rfloor$ guards are needed to guard the interior and boundary of a simple polygon with n vertices. [Hint: Use Theorem 1 in Section 5.2 to triangulate the simple polygon into $n - 2$ triangles. Then show that it is possible to color the vertices of the triangulated polygon using three colors so that no two adjacent vertices have the same color. Use induction and Exercise 23 in Section 5.2. Finally, put guards at all vertices that are colored red, where red is the color used least in the coloring of the vertices. Show that placing guards at these points is all that is needed.]

Key Terms and Results

TERMS

undirected edge: an edge associated to a set $\{u, v\}$, where u and v are vertices

directed edge: an edge associated to an ordered pair (u, v) , where u and v are vertices

multiple edges: distinct edges connecting the same vertices

multiple directed edges: distinct directed edges associated with the same ordered pair (u, v) , where u and v are vertices

loop: an edge connecting a vertex with itself

undirected graph: a set of vertices and a set of undirected edges each of which is associated with a set of one or two of these vertices

simple graph: an undirected graph with no multiple edges or loops

multigraph: an undirected graph that may contain multiple edges but no loops

pseudograph: an undirected graph that may contain multiple edges and loops

directed graph: a set of vertices together with a set of directed edges each of which is associated with an ordered pair of vertices

directed multigraph: a graph with directed edges that may contain multiple directed edges

simple directed graph: a directed graph without loops or multiple directed edges

adjacent: two vertices are adjacent if there is an edge between them

incident: an edge is incident with a vertex if the vertex is an endpoint of that edge

deg v (degree of the vertex v in an undirected graph): the number of edges incident with v with loops counted twice

chromatic number: the minimum number of colors needed in a coloring of a graph

RESULTS

The handshaking theorem: If $G = (V, E)$ be an undirected graph with m edges, then $2m = \sum_{v \in V} \deg(v)$.

Hall's marriage theorem: The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1 . There is an Euler circuit in a connected multigraph if and only if every vertex has even degree.

There is an Euler path in a connected multigraph if and only if at most two vertices have odd degree.

Dijkstra's algorithm: a procedure for finding a shortest path between two vertices in a weighted graph (see Section 10.6).

Euler's formula: $r = e - v + 2$ where r , e , and v are the number of regions of a planar representation, the number of edges, and the number of vertices, respectively, of a connected planar graph.

Kuratowski's theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5 . (Proof beyond scope of this book.)

The four color theorem: Every planar graph can be colored using no more than four colors. (Proof far beyond the scope of this book!)

Review Questions

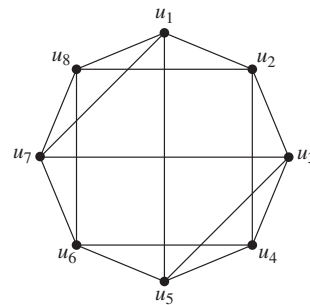
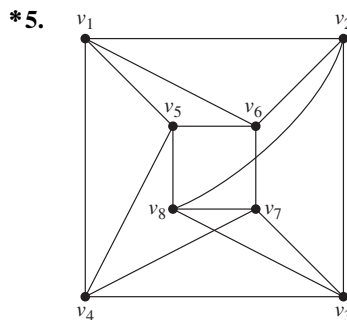
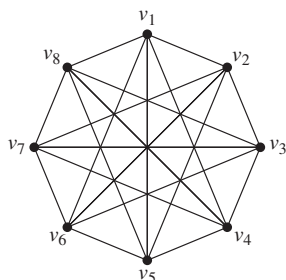
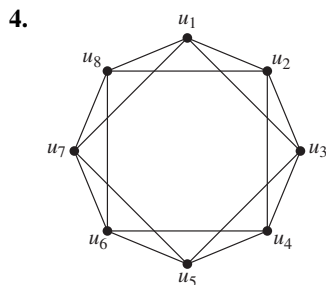
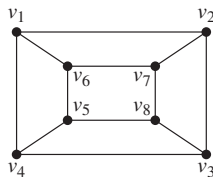
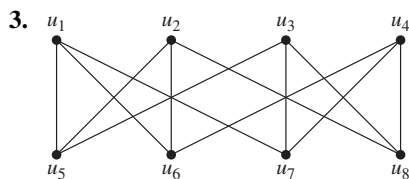
1. a) Define a simple graph, a multigraph, a pseudograph, a directed graph, and a directed multigraph.
b) Use an example to show how each of the types of graph in part (a) can be used in modeling. For example, explain how to model different aspects of a computer network or airline routes.
2. Give at least four examples of how graphs are used in modeling.
3. What is the relationship between the sum of the degrees of the vertices in an undirected graph and the number of edges in this graph? Explain why this relationship holds.
4. Why must there be an even number of vertices of odd degree in an undirected graph?
5. What is the relationship between the sum of the in-degrees and the sum of the out-degrees of the vertices in a directed graph? Explain why this relationship holds.
6. Describe the following families of graphs.
 - a) K_n , the complete graph on n vertices
 - b) $K_{m,n}$, the complete bipartite graph on m and n vertices
 - c) C_n , the cycle with n vertices
 - d) W_n , the wheel of size n
 - e) Q_n , the n -cube
7. How many vertices and how many edges are there in each of the graphs in the families in Question 6?
8. a) What is a bipartite graph?
b) Which of the graphs K_n , C_n , and W_n are bipartite?
c) How can you determine whether an undirected graph is bipartite?
9. a) Describe three different methods that can be used to represent a graph.
b) Draw a simple graph with at least five vertices and eight edges. Illustrate how it can be represented using the methods you described in part (a).
10. a) What does it mean for two simple graphs to be isomorphic?
b) What is meant by an invariant with respect to isomorphism for simple graphs? Give at least five examples of such invariants.
c) Give an example of two graphs that have the same numbers of vertices, edges, and degrees of vertices, but that are not isomorphic.
d) Is a set of invariants known that can be used to efficiently determine whether two simple graphs are isomorphic?
11. a) What does it mean for a graph to be connected?
b) What are the connected components of a graph?
12. a) Explain how an adjacency matrix can be used to represent a graph.
b) How can adjacency matrices be used to determine whether a function from the vertex set of a graph G to the vertex set of a graph H is an isomorphism?
c) How can the adjacency matrix of a graph be used to determine the number of paths of length r , where r is a positive integer, between two vertices of a graph?
13. a) Define an Euler circuit and an Euler path in an undirected graph.
b) Describe the famous Königsberg bridge problem and explain how to rephrase it in terms of an Euler circuit.
c) How can it be determined whether an undirected graph has an Euler path?
d) How can it be determined whether an undirected graph has an Euler circuit?
14. a) Define a Hamilton circuit in a simple graph.
b) Give some properties of a simple graph that imply that it does not have a Hamilton circuit.
15. Give examples of at least two problems that can be solved by finding the shortest path in a weighted graph.
16. a) Describe Dijkstra's algorithm for finding the shortest path in a weighted graph between two vertices.
b) Draw a weighted graph with at least 10 vertices and 20 edges. Use Dijkstra's algorithm to find the shortest path between two vertices of your choice in the graph.

17. a) What does it mean for a graph to be planar?
b) Give an example of a nonplanar graph.
18. a) What is Euler's formula for connected planar graphs?
b) How can Euler's formula for planar graphs be used to show that a simple graph is nonplanar?
19. State Kuratowski's theorem on the planarity of graphs and explain how it characterizes which graphs are planar.
20. a) Define the chromatic number of a graph.
- b) What is the chromatic number of the graph K_n when n is a positive integer?
- c) What is the chromatic number of the graph C_n when n is an integer greater than 2?
- d) What is the chromatic number of the graph $K_{m,n}$ when m and n are positive integers?
21. State the four color theorem. Are there graphs that cannot be colored with four colors?
22. Explain how graph coloring can be used in modeling. Use at least two different examples.

Supplementary Exercises

1. How many edges does a 50-regular graph with 100 vertices have?
2. How many nonisomorphic subgraphs does K_3 have?

In Exercises 3–5 determine whether two given graphs are isomorphic.



The **complete m -partite graph** K_{n_1, n_2, \dots, n_m} has vertices partitioned into m subsets of n_1, n_2, \dots, n_m elements each, and vertices are adjacent if and only if they are in different subsets in the partition.

6. Draw these graphs.
a) $K_{1,2,3}$ b) $K_{2,2,2}$ c) $K_{1,2,2,3}$
- *7. How many vertices and how many edges does the complete m -partite graph K_{n_1, n_2, \dots, n_m} have?
8. Prove or disprove that there are always two vertices of the same degree in a finite multigraph having at least two vertices.
9. Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that
a) $N(A \cup B) = N(A) \cup N(B)$.
b) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

10. Let $G = (V, E)$ be an undirected graph. Show that

a) $|N(v)| \leq \deg(v)$ for all $v \in V$.

b) $|N(v)| = \deg v$ for all $v \in V$ if and only if G is a simple graph.

Suppose that S_1, S_2, \dots, S_n is a collection of subsets of a set S where n is a positive integer. A **system of distinct representatives (SDR)** for this family is an ordered n -tuple (a_1, a_2, \dots, a_n) with the property that $a_i \in S_i$ for $i = 1, 2, \dots, n$ and $a_i \neq a_j$ for all $i \neq j$.

11. Find a SDR for the sets $S_1 = \{a, c, m, e\}$, $S_2 = \{m, a, c, e\}$, $S_3 = \{a, p, e, x\}$, $S_4 = \{x, e, n, a\}$, $S_5 = \{n, a, m, e\}$, and $S_6 = \{e, x, a, m\}$.

12. Use Hall's marriage theorem to show that a collection of finite subsets S_1, S_2, \dots, S_n of a set S has a SDR (a_1, a_2, \dots, a_n) if and only if $|\bigcup_{i \in I} S_i| \geq |I|$ for all subsets I of $\{1, 2, \dots, n\}$.

13. a) Use Exercise 12 to show that the collection of sets $S_1 = \{a, b, c\}$, $S_2 = \{b, c, d\}$, $S_3 = \{a, b, d\}$, $S_4 = \{b, c, d\}$ has a SDR without finding one explicitly.

b) Find a SDR for the family of four sets in part (a).

14. Use Exercise 12 to show that collection of sets $S_1 = \{a, b, c\}$, $S_2 = \{a, c\}$, $S_3 = \{c, d, e\}$, $S_4 = \{b, c\}$, $S_5 = \{d, e, f\}$, $S_6 = \{a, c, e\}$, and $S_7 = \{a, b\}$ does not have a SDR.

The **clustering coefficient** $C(G)$ of a simple graph G is the probability that if u and v are neighbors and v and w are neighbors, then u and w are neighbors, where u, v , and w are distinct vertices of G .

15. We say that three vertices u, v , and w of a simple graph G form a triangle if there are edges connecting all three pairs of these vertices. Find a formula for $C(G)$ in terms of the number of triangles in G and the number of paths of length two in the graph. [Hint: Count each triangle in the graph once for each order of three vertices that form it.]

16. Find the clustering coefficient of each of the graphs in Exercise 20 of Section 10.2

17. Explain what the clustering coefficient measures in each of these graphs.

a) the Hollywood graph

b) the graph of Facebook friends

c) the academic collaboration graph for researchers in graph theory

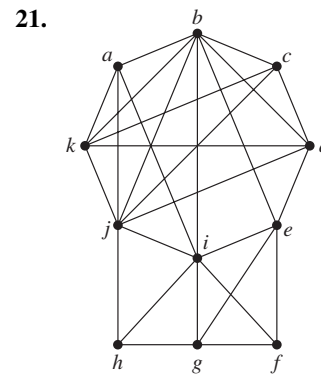
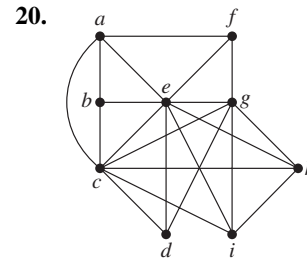
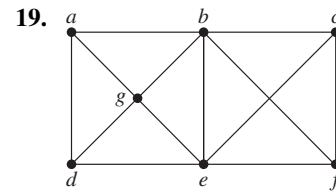
d) the protein interaction graph for a human cell

e) the graph representing the routers and communications links that make up the worldwide Internet

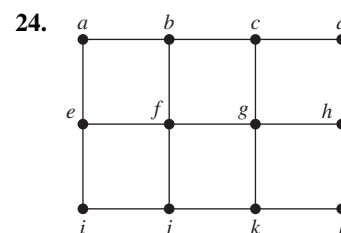
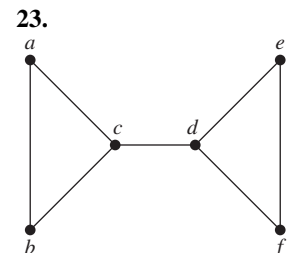
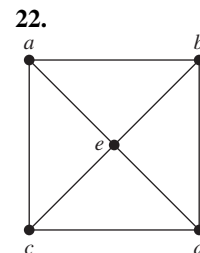
18. For each of the graphs in Exercise 17, explain whether you would expect its clustering coefficient to be closer to 0.01 or to 0.10 and why you expect this.



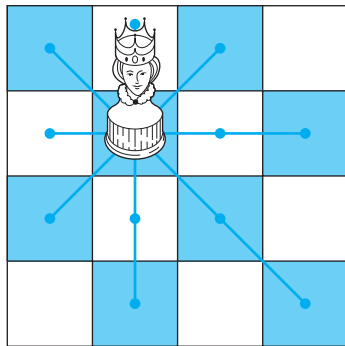
A **clique** in a simple undirected graph is a complete subgraph that is not contained in any larger complete subgraph. In Exercises 19–21 find all cliques in the graph shown.



A **dominating set** of vertices in a simple graph is a set of vertices such that every other vertex is adjacent to at least one vertex of this set. A dominating set with the least number of vertices is called a **minimum dominating set**. In Exercises 22–24 find a minimum dominating set for the given graph.



A simple graph can be used to determine the minimum number of queens on a chessboard that control the entire chessboard. An $n \times n$ chessboard has n^2 squares in an $n \times n$ configuration. A queen in a given position controls all squares in the same row, the same column, and on the two diagonals containing this square, as illustrated. The appropriate simple graph has n^2 vertices, one for each square, and two vertices are adjacent if a queen in the square represented by one of the vertices controls the square represented by the other vertex.



The Squares
Controlled
by a Queen

25. Construct the simple graph representing the $n \times n$ chessboard with edges representing the control of squares by queens for
- $n = 3$.
 - $n = 4$.
26. Explain how the concept of a minimum dominating set applies to the problem of determining the minimum number of queens controlling an $n \times n$ chessboard.
- **27.** Find the minimum number of queens controlling an $n \times n$ chessboard for
- $n = 3$.
 - $n = 4$.
 - $n = 5$.
28. Suppose that G_1 and H_1 are isomorphic and that G_2 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.
29. Show that each of these properties is an invariant that isomorphic simple graphs either both have or both do not have.
- connectedness
 - the existence of a Hamilton circuit
 - the existence of an Euler circuit
 - having crossing number C
 - having n isolated vertices
 - being bipartite
30. How can the adjacency matrix of \overline{G} be found from the adjacency matrix of G , where G is a simple graph?
31. How many nonisomorphic connected bipartite simple graphs are there with four vertices?
- *32.** How many nonisomorphic simple connected graphs with five vertices are there
- with no vertex of degree more than two?
 - with chromatic number equal to four?
 - that are nonplanar?
- A directed graph is **self-converse** if it is isomorphic to its converse.
33. Determine whether the following graphs are self-converse.
- -
34. Show that if the directed graph G is self-converse and H is a directed graph isomorphic to G , then H is also self-converse.
- An **orientation** of an undirected simple graph is an assignment of directions to its edges such that the resulting directed graph is strongly connected. When an orientation of an undirected graph exists, this graph is called **orientable**. In Exercises 35–37 determine whether the given simple graph is orientable.
- -
 -

38. Because traffic is growing heavy in the central part of a city, traffic engineers are planning to change all the streets, which are currently two-way, into one-way streets. Explain how to model this problem.
- *39. Show that a graph is not orientable if it has a cut edge.
A **tournament** is a simple directed graph such that if u and v are distinct vertices in the graph, exactly one of (u, v) and (v, u) is an edge of the graph.
40. How many different tournaments are there with n vertices?
41. What is the sum of the in-degree and out-degree of a vertex in a tournament?
- *42. Show that every tournament has a Hamilton path.
43. Given two chickens in a flock, one of them is dominant. This defines the **pecking order** of the flock. How can a tournament be used to model pecking order?
44. Suppose that a connected graph G has n vertices and vertex connectivity $\kappa(G) = k$. Show that G must have at least $\lceil kn/2 \rceil$ edges.
- A connected graph $G = (V, E)$ with n vertices and m edges is said to have **optimal connectivity** if $\kappa(G) = \lambda(G) = \min_{v \in V} \deg v = 2m/n$.
45. Show that a connected graph with optimal connectivity must be regular.
46. Show these graphs have optimal connectivity.
a) C_n for $n \geq 3$
b) K_n for $n \geq 3$
c) $K_{r,r}$ for $r \geq 2$
- *47. Find the two nonisomorphic simple graphs with six vertices and nine edges that have optimal connectivity.
48. Suppose that G is a connected multigraph with $2k$ vertices of odd degree. Show that there exist k subgraphs that have G as their union, where each of these subgraphs has an Euler path and where no two of these subgraphs have an edge in common. [Hint: Add k edges to the graph connecting pairs of vertices of odd degree and use an Euler circuit in this larger graph.]

In Exercises 49 and 50 we consider a puzzle posed by Petković in [Pe09] (based on a problem in [AvCh80]). Suppose that King Arthur has gathered his $2n$ knights of the Round Table for an important council. Every two knights are either friends or enemies, and each knight has no more than $n - 1$ enemies among the other $2n - 1$ knights. The puzzle asks whether King Arthur can seat his knights around the Round Table so that each knight has two friends for his neighbors.

49. a) Show that the puzzle can be reduced to determining whether there is a Hamilton circuit in the graph in which each knight is represented by a vertex and two knights are connected in the graph if they are friends.
b) Answer the question posed in the puzzle. [Hint: Use Dirac's theorem.]
50. Suppose that are eight knights Alynore, Bedivere, Degore, Gareth, Kay, Lancelot, Perceval, and Tristan. Their

lists of enemies are A (D, G, P), B (K, P, T), D (A, G, L), G (A, D, T), K (B, L, P), L (D, K, T), P (A, B, K), T (B, G, L), where we have represented each knight by the first letter of his name and shown the list of enemies of that knight following this first letter. Draw the graph representing these eight knight and their friends and find a seating arrangement where each knight sits next to two friends.

- *51. Let G be a simple graph with n vertices. The **bandwidth** of G , denoted by $B(G)$, is the minimum, over all permutations a_1, a_2, \dots, a_n of the vertices of G , of $\max\{|i - j| \mid a_i \text{ and } a_j \text{ are adjacent}\}$. That is, the bandwidth is the minimum over all listings of the vertices of the maximum difference in the indices assigned to adjacent vertices. Find the bandwidths of these graphs.

- a) K_5 b) $K_{1,3}$ c) $K_{2,3}$
d) $K_{3,3}$ e) Q_3 f) C_5

- *52. The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2 . The **radius** of a graph is the minimum over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

- a) K_6 . b) $K_{4,5}$. c) Q_3 . d) C_6 .

- *53. a) Show that if the diameter of the simple graph G is at least four, then the diameter of its complement \overline{G} is no more than two.
b) Show that if the diameter of the simple graph G is at least three, then the diameter of its complement \overline{G} is no more than three.

- *54. Suppose that a multigraph has $2m$ vertices of odd degree. Show that any circuit that contains every edge of the graph must contain at least m edges more than once.


55. Find the second shortest path between the vertices a and z in Figure 3 of Section 10.6.

56. Devise an algorithm for finding the second shortest path between two vertices in a simple connected weighted graph.

57. Find the shortest path between the vertices a and z that passes through the vertex f in the weighted graph in Exercise 3 in Section 10.6.

58. Devise an algorithm for finding the shortest path between two vertices in a simple connected weighted graph that passes through a specified third vertex.

- *59. Show that if G is a simple graph with at least 11 vertices, then either G or \overline{G} , the complement of G , is nonplanar.

 A set of vertices in a graph is called **independent** if no two vertices in the set are adjacent. The **independence number** of a graph is the maximum number of vertices in an independent set of vertices for the graph.

- *60. What is the independence number of

- a) K_n ? b) C_n ? c) Q_n ? d) $K_{m,n}$?

61. Show that the number of vertices in a simple graph is less than or equal to the product of the independence number and the chromatic number of the graph.
62. Show that the chromatic number of a graph is less than or equal to $n - i + 1$, where n is the number of vertices in the graph and i is the independence number of this graph.
63. Suppose that to generate a random simple graph with n vertices we first choose a real number p with $0 \leq p \leq 1$. For each of the $C(n, 2)$ pairs of distinct vertices we generate a random number x between 0 and 1. If $0 \leq x \leq p$, we connect these two vertices with an edge; otherwise these vertices are not connected.
 - a) What is the probability that a graph with m edges where $0 \leq m \leq C(n, 2)$ is generated?
 - b) What is the expected number of edges in a randomly generated graph with n vertices if each edge is included with probability p ?
 - c) Show that if $p = 1/2$ then every simple graph with n vertices is equally likely to be generated.

A property retained whenever additional edges are added to a simple graph (without adding vertices) is called **monotone increasing**, and a property that is retained whenever edges are

removed from a simple graph (without removing vertices) is called **monotone decreasing**.

64. For each of these properties, determine whether it is monotone increasing and determine whether it is monotone decreasing.
 - a) The graph G is connected.
 - b) The graph G is not connected.
 - c) The graph G has an Euler circuit.
 - d) The graph G has a Hamilton circuit.
 - e) The graph G is planar.
 - f) The graph G has chromatic number four.
 - g) The graph G has radius three.
 - h) The graph G has diameter three.
65. Show that the graph property P is monotone increasing if and only if the graph property Q is monotone decreasing where Q is the property of not having property P .
- **66. Suppose that P is a monotone increasing property of simple graphs. Show that the probability a random graph with n vertices has property P is a monotonic nondecreasing function of p , the probability an edge is chosen to be in the graph.

Computer Projects

Write programs with these input and output.

1. Given the vertex pairs associated to the edges of an undirected graph, find the degree of each vertex.
2. Given the ordered pairs of vertices associated to the edges of a directed graph, determine the in-degree and out-degree of each vertex.
3. Given the list of edges of a simple graph, determine whether the graph is bipartite.
4. Given the vertex pairs associated to the edges of a graph, construct an adjacency matrix for the graph. (Produce a version that works when loops, multiple edges, or directed edges are present.)
5. Given an adjacency matrix of a graph, list the edges of this graph and give the number of times each edge appears.
6. Given the vertex pairs associated to the edges of an undirected graph and the number of times each edge appears, construct an incidence matrix for the graph.
7. Given an incidence matrix of an undirected graph, list its edges and give the number of times each edge appears.
8. Given a positive integer n , generate a simple graph with n vertices by producing an adjacency matrix for the graph so that all simple graphs with n vertices are equally likely to be generated.
9. Given a positive integer n , generate a simple directed graph with n vertices by producing an adjacency matrix for the graph so that all simple directed graphs with n vertices are equally likely to be generated.
10. Given the lists of edges of two simple graphs with no more than six vertices, determine whether the graphs are isomorphic.
11. Given an adjacency matrix of a graph and a positive integer n , find the number of paths of length n between two vertices. (Produce a version that works for directed and undirected graphs.)
- *12. Given the list of edges of a simple graph, determine whether it is connected and find the number of connected components if it is not connected.
13. Given the vertex pairs associated to the edges of a multigraph, determine whether it has an Euler circuit and, if not, whether it has an Euler path. Construct an Euler path or circuit if it exists.
- *14. Given the ordered pairs of vertices associated to the edges of a directed multigraph, construct an Euler path or Euler circuit, if such a path or circuit exists.
- **15. Given the list of edges of a simple graph, produce a Hamilton circuit, or determine that the graph does not have such a circuit.
- **16. Given the list of edges of a simple graph, produce a Hamilton path, or determine that the graph does not have such a path.
17. Given the list of edges and weights of these edges of a weighted connected simple graph and two vertices in this graph, find the length of a shortest path between them using Dijkstra's algorithm. Also, find a shortest path.