

Chapter 2

Stress Definition

“Stress tensor” is a tautology since the term tensor (latin tensio) means stress. The pleonasm, however, became naturalized in books dealing with mechanical stress (Hahn 1985, p. 20, fn. 1). Referring to other second-rank tensors in physics like the inertial tensor; this tautology disappears

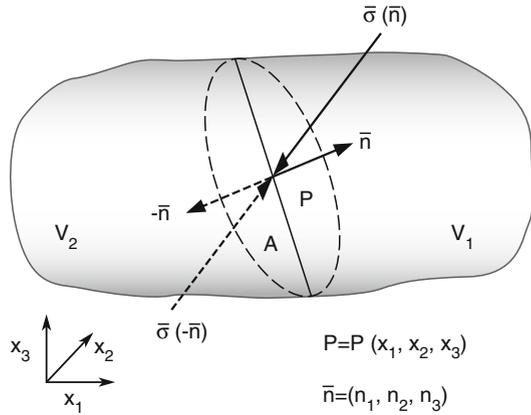
This chapter presents the fundamental concept of stress as it is defined from a mathematical, physical and continuum mechanics point of view. The stress tensor defining the state of stress at a point is introduced using the continuum concept of a stress vector (traction) defining the state of stress on a plane (Sect. 2.1). Principal stresses and their orientations are deduced from solving the eigenvalue problem (Sect. 2.2). The Mohr circle of stress is a way of visualizing normal and shear stress components for traction vectors associated with all possible planes through one point (Sect. 2.3). Since elastic stress is a fictitious term, the display of stress involves some mathematical gimmicks (Sect. 2.4).

2.1 Stress Tensor



In this section, mechanical stress is quantified mathematically as a second-order tensor and physically by its tensor invariants. In analogy to continuum mechanics (Fung 1965; Timoshenko and Goodier 1970; Hahn 1985), consider a deformable body subjected to some arbitrary sets of loads in equilibrium (Fig. 2.1). At any given point $P(\bar{x}) = P(x_1, x_2, x_3)$ within this body, we imagine a plane A slicing through the body at an angle with respect to the Cartesian coordinate system with unit vectors $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$. The fictitious slicing plane (Sect. 1.1) divides the body into volumes V_1 and V_2 , and has a normal $\bar{n} = (n_1, n_2, n_3)$ which points towards V_1 . The action that V_1 exerts on V_2 is denoted by a resultant force $\bar{F} = (F_1, F_2, F_3)$. The *traction vector* $\bar{\sigma}$ is defined as the ratio of the resultant force \bar{F} to the surface area A (Fig. 2.1). In order to define the traction that acts over a specific point $P(\bar{x})$ in the body, the area A is now allowed to contract to a point ($dA \rightarrow 0$), so that the magnitude A goes to zero.

Fig. 2.1 Traction vector $\bar{\sigma}$ acting on a hypothetical (fictitious) slicing plane A with surface normal \bar{n} within a deformable body



$$\bar{\sigma}(P(\bar{x}), \bar{n}) = \lim_{\Delta A \rightarrow 0} \frac{d\bar{F}}{dA} \quad (2.1)$$

In general, the traction vector $\bar{\sigma}$ can vary from point to point, and is therefore a function of the location of the point $P(\bar{x})$. However, at any given point, the traction will also, in general, be different on different planes that pass through the point. Therefore, $\bar{\sigma}$ will also be a function of \bar{n} , the outward unit normal vector of the slicing plane. In summary, $\bar{\sigma}$ is a function of two vectors, the position vector \bar{x} and the normal vector of the slicing plane \bar{n} . In 1823, the French mathematician *Augustin Baron Cauchy* (1789–1857) introduced the concept of stress by eliminating the difficulty that $\bar{\sigma}$ is a function of two vectors, $\bar{\sigma}(\bar{x}, \bar{n})$ at the price that stress became a second-order tensor (Jaeger et al. 2007).

We have three remarks about Eq. (2.1). First, Eq. (2.1) is an empirical formula, i.e. is confirmed by experimental findings. Second, there are obvious practical limitations in reducing the size of a small area to zero, but it is important that, formally, *the stress* is defined in this way as a *point property*. Third, the magnitude of the total traction vector is

$$|\bar{\sigma}(P(\bar{x}), \bar{n})| = \frac{dF}{dA}. \quad (2.2)$$

To uniquely identify stress as a second-order tensor, Cauchy verified two laws. *Cauchy's first law* is visualized in Fig. 2.1 and reads

$$\bar{\sigma}(-\bar{n}) = -\bar{\sigma}(\bar{n}). \quad (2.3)$$

Equation (2.3) is a version of Newton's third law "*actio = reactio*" we know from Sect. 1.1. If material to the right of the slicing plane (Fig. 2.1, volume V_1) exerts a traction $\bar{\sigma}$ on the material to the left (Fig. 2.1, volume V_2), then the material to the

left will exert a traction $-\bar{\sigma}$ on the material to the right. The Cartesian component of the traction vector in any given direction is considered to be positive if the inner product (dot product) is negative. This is called the *rock mechanics sign convention* (compression positive), and is inconsistent with most areas of mechanics where tension positive convention is used.

Cauchy's second law states that all possible traction vectors at a point (infinite number) corresponding to all possible slicing planes passing through that point (infinite number), can be found from the knowledge of traction vector on three mutually orthogonal planes in 3D. To derive this relationship for the traction on an arbitrary plane, Cauchy introduced an infinitesimal tetrahedron (Fig. 2.2). In the *Cauchy tetrahedron*, the arbitrary slicing plane dA is chosen as a small inclined triangle close to the point $P(\bar{x}) = P(0, 0, 0)$ at which the state of stress needs to be known. Cauchy's second law can be derived from balancing forces at the tetrahedron (Fig. 2.2). Three faces of the tetrahedron have outward unit normal vectors that coincide with the negative Cartesian coordinate directions ($-\bar{e}_1 = (-1, 0, 0)$, $-\bar{e}_2 = (0, -1, 0)$, $-\bar{e}_3 = (0, 0, -1)$). The inclined face of the tetrahedron has an outward unit normal vector of

$$\bar{n} = (n_1, n_2, n_3) = \cos(\bar{n}, \bar{e}_i). \tag{2.4}$$

The components of the vector \bar{n} are given by the direction cosines that the outward unit normal vector of the fourth face makes with the three Cartesian coordinate axes. As the length of any unit vector is unity, $n_1^2 + n_2^2 + n_3^2 = 1$ applies. The area of the face with unit vector \bar{n} is taken to be dA . The areas of the three other faces with outward unit normal vectors $\bar{n} = -\bar{e}_i$ equal

$$dA_i = n_i dA. \tag{2.5}$$

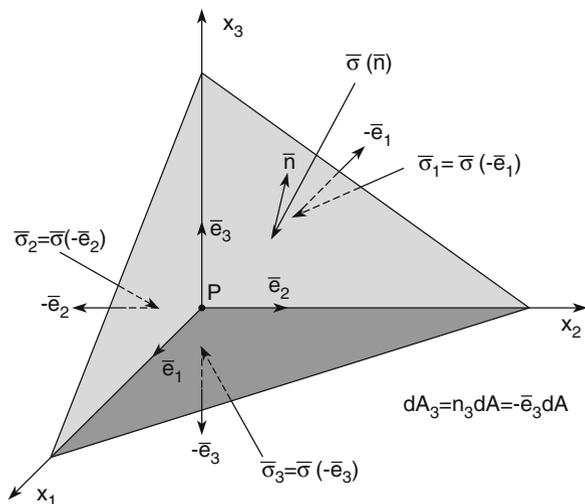


Fig. 2.2 Cauchy tetrahedron with traction vector balance on three Cartesian planes and the inclined slicing plane close to the origin of the infinitesimal tetrahedron

The traction vectors on these faces are denoted by (Fig. 2.2)

$$\bar{\sigma}_i = \bar{\sigma}(-\bar{e}_i) \quad (2.6)$$

and so the total force acting on these face are

$$dA_i \bar{\sigma}_i = n_i dA \bar{\sigma}(-\bar{e}_i). \quad (2.7)$$

Balancing forces on the inclined face of the tetrahedron leads to

$$\bar{\sigma}(\bar{n})dA + n_1 dA \bar{\sigma}(-\bar{e}_1) + n_2 dA \bar{\sigma}(-\bar{e}_2) + n_3 dA \bar{\sigma}(-\bar{e}_3) = 0. \quad (2.8)$$

Cancelling out the common area dA , and utilizing Cauchy's first law, for example $\bar{\sigma}(-\bar{e}_1) = -\bar{\sigma}(\bar{e}_1)$, leads to *Cauchy's second law*:

$$\begin{aligned} \bar{\sigma}(\bar{n}) &= n_1 \bar{\sigma}(\bar{e}_1) + n_2 \bar{\sigma}(\bar{e}_2) + n_3 \bar{\sigma}(\bar{e}_3) \\ \sigma_i(n_j) &= \sigma_{ij}^T n_j \end{aligned} \quad (2.9)$$

It is written both in vector notation ((2.9), upper half) and index notation ((2.9), lower half). The components of the three traction vectors that act on planes whose outward unit normals are in the three coordinate directions are denoted by

$$\begin{aligned} \bar{\sigma}(\bar{e}_1) &= (\sigma_{11}, \sigma_{12}, \sigma_{13})^T \\ \bar{\sigma}(\bar{e}_2) &= (\sigma_{21}, \sigma_{22}, \sigma_{23})^T, \\ \bar{\sigma}(\bar{e}_3) &= (\sigma_{31}, \sigma_{32}, \sigma_{33})^T \end{aligned} \quad (2.10)$$

where T stands for the transpose of a row vector since traction vector components appear as columns in Eq. (2.9). The components of each traction vector are denoted by two indices. The first refers to the direction of the outward unit normal vector \bar{e}_i and the second refers to the component of the traction vector $\bar{\sigma}$. Substituting (2.10) into (2.9) leads to

$$\begin{aligned} \bar{\sigma}_1(\bar{n}) &= \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 \\ \bar{\sigma}_2(\bar{n}) &= \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3. \\ \bar{\sigma}_3(\bar{n}) &= \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3 \end{aligned} \quad (2.11)$$

If one uses the standard matrix algebraic convention that the first subscript of the matrix components denotes the row and the second subscript denotes the column, it follows that

$$\begin{pmatrix} \sigma_1(\bar{n}) \\ \sigma_2(\bar{n}) \\ \sigma_3(\bar{n}) \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \quad (2.12)$$

The matrix appearing in (2.12) is the transpose of the stress matrix. The *stress tensor* mathematically given by the stress matrix unequivocally defines the state of stress at an arbitrary point within a deformable body.

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad (2.13)$$

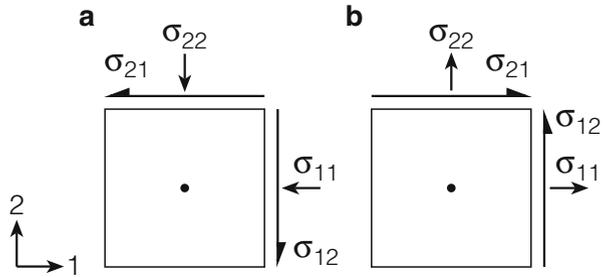
In 3D the stress tensor has nine components. The rows of the stress tensor are the traction vectors along the coordinate axes. Equation (2.12) is usually written with the transpose matrix defined in Eq. (2.13), because the stress matrix is always symmetric

$$\begin{aligned} \bar{\sigma} &= \sigma_{ij}\bar{n} = \sigma_{ij}^T\bar{n} \\ \sigma_i &= \sigma_{ij}n_j \equiv \sigma_{ji}n_j \end{aligned} \quad (2.14)$$

Again, the equation is written in both vector notation ((2.14), upper half) and index notation ((2.14), lower half). The symmetry of the stress tensor can be proven by applying the mechanical law of conservation of angular momentum. The symmetry of the stress tensor reduces the components from nine to six in Eq. (2.13). Based on this property of the stress tensor, the first subscript i in (2.13), can be specified as normal to the actual slicing surface, while the second subscript j in (2.13), can be identified with the direction of the force. Depending upon the orientation of the slicing surface being normal and the force we can distinguish *normal stress* ($i=j$, force perpendicular slicing plane) with components $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ pointing towards Cartesian axes and *shear stress* ($i \neq j$, force parallel slicing plane) with components $(\sigma_{12} = \sigma_{21}, \sigma_{13} = \sigma_{31}, \sigma_{23} = \sigma_{32})$ effective within Cartesian planes.

The physical significance of the stress tensor is illustrated by a 2D square element of an elastic body in Fig. 2.3. The *rock mechanics sign convention* is illustrated in Fig. 2.3a. The traction vector that acts on the face whose outward unit normal vector is in the x_1 direction, has components $(\sigma_{11}, \sigma_{12})$. As the traction components are considered positive if they are oriented in the directions opposite to the outward unit normal vector (Fig. 2.3a), we see that the traction σ_{11} is a positive number if it is compressive (Fig. 2.3a). In Fig. 2.3b, the *engineering mechanics sign convention* is illustrated with tensile normal stresses treated as positive. The direction of positive shear stresses is as shown in both sign conventions. Since compressive stresses are much more common for rocks in the Earth's crust, the rock mechanics sign convention (compression positive) is more appropriate in order to avoid frequent occurrence of negative signs in calculations involving stresses.

Fig. 2.3 Visualization of stress components in 2D. Sign convention in **a** rock mechanics or geosciences (compression positive), and **b** engineering mechanics or material sciences (tension positive)



We have three final remarks about the stress tensor. Firstly, the stress tensor can be calculated from the inner product (dot product) of the traction vectors of the unit vectors of the Cartesian reference frame (see (2.13) and (2.14), $\sigma_{ij} = \bar{e}_j \cdot \bar{\sigma}_i$). Secondly, the stress tensor can be written as a matrix, but a tensor has specific physical properties that are more important compared to that of a regular matrix. These properties relate to the manner in which the components of a tensor transform when the coordinate system is changed (Sect. 2.2). Thirdly, the fact that the state of stress at a point in 3D is completely specified by six independent components is important for the stress measuring techniques discussed in Part II of the book (Chaps. 6–8).

The unit of stress is pascal $\text{Pa} = \text{Nm}^{-2}$. The stress magnitude 1 Pa is produced by the force 1 N which acts normal or parallel to a square metre large surface. As 1 N is a small force and 1 m^2 a large surface, 1 Pa is a very small stress (= force/area). The “old” unit atmospheric pressure (1 bar) corresponds to 100 kPa. Crustal stresses in the Earth are usually measured in mega pascal whereby 1 MPa (“new unit”) equals 10 bars (“old unit”). The stress magnitude 1 MPa is equal to the pressure p at a depth z of about 100 m in water, or about 37 m in rock using the relationship $p = \rho g z$, where ρ is the density of material (1000 kg m^{-3} for water, 2700 kg m^{-3} for rock) and g is the acceleration due to gravity, 9.81 m s^{-2} . Note that stress is not the same as pressure. Pressure is reserved for a specific stress state in which there are no shear components and all normal components are equal (e.g., in a fluid).

Note-Box A specific stress component acting on a specific slicing plane inside a deformable body can be described by a stress vector (traction). Three traction vectors are needed to unequivocally define the state of stress at a point inside the body resulting in nine components of the physical quantity stress tensor. Mathematically, the stress tensor can be written as stress matrix representing all stress components acting on three orthogonal slicing planes through a single, arbitrarily chosen body point. Due to the symmetry, only six stress components remain independent in the stress tensor (three normal and three shear stresses). The stress unit is force per area $\text{N m}^{-2} = \text{Pa}$ (pascal), whereby 1 MPa equals 10 bar.

2.2 Principal Stresses

The principal stresses and principal directions can be found by asking whether or not there are planes on which the traction vector is purely normal, with no shear component. On such planes, the traction vector is aligned parallel to the outward unit vector, and can therefore be expressed using Cauchy's second law (see Eq. (2.14)) as

$$\begin{aligned}\bar{\sigma} &= \sigma \bar{n} \\ \sigma_i &= \sigma_{ij} n_j = \sigma n_j\end{aligned}\quad (2.15)$$

where σ is a yet unknown scalar quantity. With help of *Kronecker's delta* ($\delta_{ij} = 1$ for $i=j$ and $\delta_{ij} = 0$ for $i \neq j$) it follows that

$$(\sigma_{ij} - \sigma \delta_{ij}) n_j = 0. \quad (2.16)$$

This is the fundamental equation for determining *eigenvalues* (principal stress magnitudes) and *eigenvectors* (principal stress directions) of the stress matrix. Splitting it into components results in the following set of equations

$$\begin{aligned}(\sigma_{11} - \sigma) n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 &= 0 \\ \sigma_{21} n_1 + (\sigma_{22} - \sigma) n_2 + \sigma_{23} n_3 &= 0. \\ \sigma_{31} n_1 + \sigma_{32} n_2 + (\sigma_{33} - \sigma) n_3 &= 0\end{aligned}\quad (2.17)$$

Admissible solutions of this linear, homogeneous set of equations can be found only if the determinant of the matrix coefficients equals zero, i.e.

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{vmatrix} \stackrel{!}{=} 0. \quad (2.18)$$

When the determinant is expanded out, it takes the form of a cubic equation in σ

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0, \quad (2.19)$$

where (I_1, I_2, I_3) are called *tensor invariants*. Values of stress invariants are independent of the coordinate system used. The physical content of a stress tensor is reflected exclusively in the stress invariants. For example, pressure in all directions, as is the case in the hydrostatic state of stress, results from I_1 . The three invariants of the stress tensor are given by

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \text{trace}(\sigma_{ij})$$

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \quad (2.20)$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} = \frac{1}{6}(\sigma_{ii}\sigma_{jj}\sigma_{kk} + 2\sigma_{ij}\sigma_{ik}\sigma_{ki} - 3\sigma_{ij}\sigma_{ij}\sigma_{kk})$$

I_1 is called the *trace* of the stress tensor and $I_{2,3}$ can be computed from the coefficient determinants. The three solutions of the characteristic Eq. (2.19) are called *principal normal stresses* ($\sigma_1, \sigma_2, \sigma_3$). The double subscripts of the normal stress components can be reduced to single suffixes, since the shear components per definition become zero. With the help of the fundamental theorem of algebra, Eq. (2.19) can be written as

$$(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3) = 0. \quad (2.21)$$

Then the tensor invariants follow from principal normal stresses

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1. \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \quad (2.22)$$

After the transformation, the stress matrix has the following (diagonal) form:

$$\sigma_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \quad (2.23)$$

whereby the principal axes are chosen in a way that magnitudes $\sigma_1 \geq \sigma_2 \geq \sigma_3$ apply for the principal stress. Using the principal stress magnitudes, the direction cosines of the principal axes can be deduced from Eq. (2.4).

The principal normal stress components can be visualized in 3D using a rotated cube (Fig. 2.4). Principal stress directions (x', y', z') are rotated with respect to the global (space-fixed) coordinate system (x, y, z). While six components of stress (taking into account the symmetry of stress tensor) are necessary to define the state of stress in an arbitrary oriented cube (Fig. 2.4a), only three components of stress ($\sigma_1, \sigma_2, \sigma_3$) are required in the rotated cube of principal stresses (Fig. 2.4b). Due to the fact that besides the magnitudes of the three principal stresses (Fig. 2.4b, eigenvalues $\sigma_1, \sigma_2, \sigma_3$) also the directions of the three principal stresses (Fig. 2.4b, eigenvectors

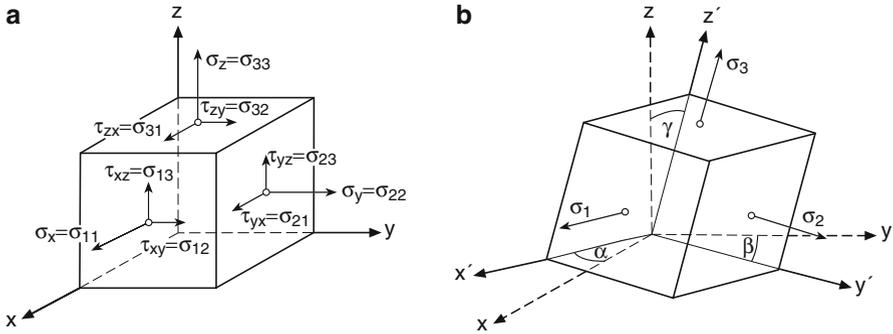


Fig. 2.4 Visualization of stress components in 3D on a cube **a** before and **b** after solving the eigenvalue problem of the stress matrix; engineering mechanics notation (tension positive convention)

α, β, γ) are necessary, six pieces of information are required for full determination of the state of stress at any point in any coordinate system. Principal normal stresses act perpendicular to the faces of the cube, the surfaces of which are free of shear stresses. An additional exercise for the reader is to redraw Fig. 2.4 so that it describes the situation when using rock mechanics (compression positive convention).

It is useful to have a way of presenting the stress tensor that clearly shows whether or not there are any shear stresses acting at the point in question. To do so, the stress tensor is decomposed into an isotropic (= hydrostatic) and a deviatoric part. The *isotropic part* of the stress tensor is defined as

$$\begin{aligned} \sigma_{ij}^{iso} &= \frac{1}{3} I_1 I = \sigma_m I \\ \sigma_m &= \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \end{aligned} \quad (2.24)$$

where I is the *identity tensor* ($I\bar{a} = \bar{a}$) and σ_m is the *mean normal stress*. The *deviatoric stress* is obtained by subtracting the isotropic part of the stress tensor from the full stress tensor.

$$\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{iso} = \begin{pmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_m \end{pmatrix}. \quad (2.25)$$

The usefulness of this decomposition arises from the fact that, in the elastic range of deformation, the isotropic stress controls the volumetric change of a body, whereas the deviatoric stress controls the distortion. Even at very high levels of stress, no plastic flow is caused by a hydrostatic stress, because there are no shear stresses on any plane, since all planes are principal planes. The deviatoric stress, however, produces shear stress and can therefore lead to plastic flow if the elastic limit of material is exceeded. A deviatoric stress causes no dilatation because the sum of its components is always zero. Rock failure criteria (Chap. 3) are concerned primarily with distortion, in which case these criteria are most conveniently expressed in

terms of the invariants of the stress deviation. J_2 is the *invariant of the stress deviator* that appears most often in rock failure criteria.

$$\begin{aligned} J_2 &= \frac{1}{2}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 \\ J_2 &= \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]. \quad (2.26) \\ J_2 &= 3\sigma_m^2 + I_2 = \frac{3}{2}\tau_{OCT}^2 \end{aligned}$$

Apart from principal normal stresses, it should be mentioned that also *principal shear stresses* exist which act in planes which are parallel to one principal axis and form an angle of 45° with the two other principal axes. To the magnitudes of principal shear stresses applies

$$\tau_1 + \tau_2 + \tau_3 = \frac{\sigma_2 - \sigma_3}{2} + \frac{\sigma_3 - \sigma_1}{2} + \frac{\sigma_1 - \sigma_2}{2} = 0. \quad (2.27)$$

With $\sigma_1 \geq \sigma_2 \geq \sigma_3$, where σ_1 is the maximum and σ_3 is the minimum (least) principal normal stress component, the *maximum shear stress* results in

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = |\tau_2|. \quad (2.28)$$

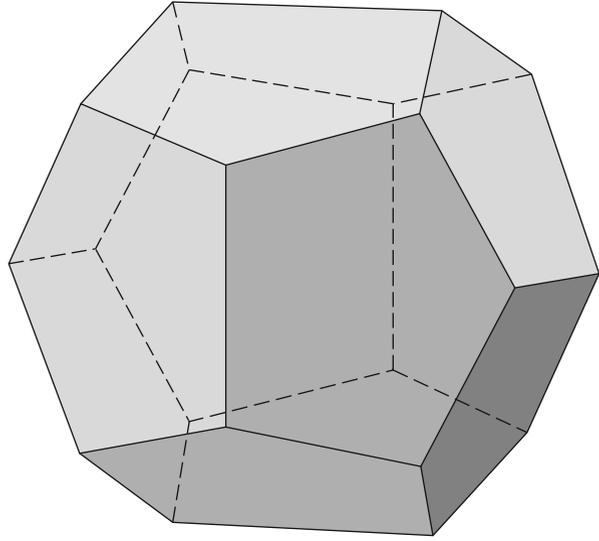
The maximum shear stress acts in a plane which cuts the angle between maximum and minimum principal normal stress into half. The planes in which the principal shear stresses act are not perpendicular to each other. They form a normal dodecahedron (Fig. 2.5). The planes of the dodecahedron are not free of normal stresses. Shear stress values are dictated by (2.27).

Exercise 2.1 Transformation of a (3×3) stress matrix. Consider the stress matrix

$$\sigma_{ij} = \begin{pmatrix} \left(\frac{13}{4}\right) & \left(\frac{\sqrt{3}}{4}\right) & 0 \\ \left(\frac{\sqrt{3}}{4}\right) & \left(\frac{15}{4}\right) & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

- Find the three principal stresses ($\sigma_1, \sigma_2, \sigma_3$) by solving the eigenvalue problem.
- Compute the stress invariants (I_1, I_2, I_3) from principal stresses.
- What are the direction cosines of the planes on which the principal stresses act?
- The matrix formed by the nine components of the three eigenvectors describes what kind of geometrical operation in space?

Fig. 2.5 Visualization of planes in which principal shear stresses are effective on a pentagon dodecahedron (3D arrangement of 12 surfaces with 5 corners)



We have two final remarks about principal stresses. Firstly, the fact that the stresses transform in an eigenvalue problem according to $\sigma'_{ij} = R\sigma_{ij}R^T$ when the coordinate system is rotated is the defining property that makes the stress a *second-order tensor*. The traction vector transforms according to $\bar{\sigma}' = R\bar{\sigma}$. The appearance of only one rotation matrix, R in this transformation law is the reason that vectors are referred to as *first-order tensors*. A *zero-order tensor* is a scalar quantity with magnitude only (e.g. temperature). Secondly, principal stresses have particular significance for rock engineering. The process of creating a new surface in a rock mass by excavation causes principal stresses to be locally oriented perpendicular and parallel to the free surface. The principal stress perpendicular to the free surface is zero. The other two principal stresses, i.e. the maximum and minimum value, occur in a direction parallel to the rock-free surface. Therefore, any excavation plane within the Earth's crust is a principal stress plane.

Note-Box Each stress matrix can be transformed from an arbitrary reference frame into the frame of principal axes. The state of stress in the new system is then defined by three principal stresses (stress magnitudes) and three principal axes (stress orientations). The physical meaning of the stress tensor is captured in tensor invariants, which are independent of the reference frame used. Principal normal stresses are visualized on a cube the faces of which are free of shear stresses. Principal shear stresses exist and act on a dodecahedron, which planes are not free of normal stresses. All unsupported rock excavation surfaces within the Earth's crust are principal normal stress planes.

2.3 Mohr Circle of Stress

To make practical use of the stress matrix in Eq. (2.13), we must be able to find the stress components in directions different from the reference directions. One possibility was demonstrated by the principal stress matrix in Eq. (2.23). The Mohr circle of stress is a second, simple graphical method of transforming the stress tensor. As the discussion of stress is algebraically simpler in 2D than in 3D, the Mohr circle is introduced in 2D. Many problems in rock mechanics are essentially 2D as stresses do not vary along one Cartesian coordinate perpendicular to the free surface (cf. stresses around boreholes, Chap. 7). Hence, it is worthwhile to study the properties of 2D stress tensors.

Consider the arbitrary plane ds (unit length assumed in the z -direction) in a deformable body whose normal makes an angle α with the orientation of maximum principal stress σ_1 (Fig. 2.6). We look for the normal stress (σ) and shear stress component (τ) acting on the surface element ds as a function of principal stresses and tilt angle ($\sigma_1, \sigma_2, \alpha$). Again, two equilibrium conditions must be fulfilled. For infinitesimal volumes, the balance of moments leads to the condition that pairs of shear stresses must be equal (symmetry of stress tensor). The balance of forces prevents the prism from translation and rotation. Note that in the 2D description of the Mohr circle, we have to operate with prisms (Fig. 2.6, triangles with unit length in the third direction), since stress is defined as force per unit area.

The balance of forces (force = stress times area) at the small prism surface ds (area = length (ds) times unit length (1)) reads

$$\begin{aligned} \sigma_2 \sin \alpha ds - \sigma \sin \alpha ds + \tau \cos \alpha ds &= 0 \\ \sigma_1 \cos \alpha ds - \sigma \cos \alpha ds - \tau \sin \alpha ds &= 0 \end{aligned} \tag{2.29}$$

Using trigonometric identities

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\ \sin 2\alpha &= 2 \sin \alpha \cos \alpha \end{aligned} \tag{2.30}$$

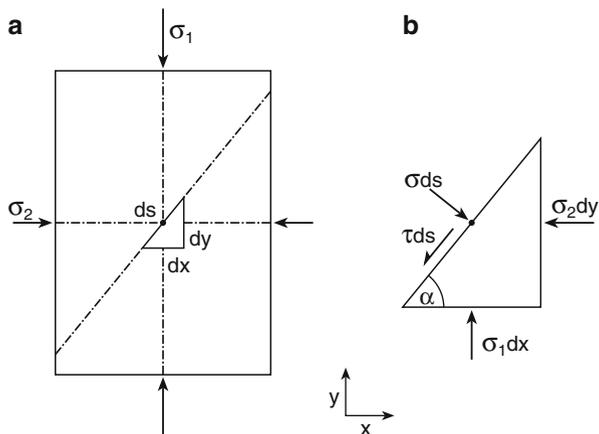


Fig. 2.6 In a deformable body loaded by minimum and maximum principal stress **a** a small prism is shown **b** where the balance of forces is computed

the normal stress σ and shear stress τ on the prism surface ds follow as a function of the two principal stresses

$$\begin{aligned} \sigma &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\alpha \\ \tau &= \frac{\sigma_1 - \sigma_2}{2} \sin 2\alpha \end{aligned} \quad (2.31)$$

These are the equations of a circle in the (σ, τ) plane with its centre at the point $(\sigma = (\sigma_1 + \sigma_2)/2, \tau = 0)$ and with radius $(\sigma_1 - \sigma_2)/2$. Equation (2.31) defines the *Mohr circle of stress*. This representation of stress was first proposed in 1882 by the German engineer Otto Mohr (1835–1918). It relates the principal stresses applied to a deformable body to the normal and shear stresses of an arbitrary oriented plane inside the body (Fig. 2.6). Equations (2.29)–(2.31) express the crucial difference between forces and stresses, which is the key to understanding the concept of stress. The resolution of a normal force requires, e.g., $\cos\alpha$, the resolution of a normal stress component however requires $\cos^2\alpha$. One $\cos\alpha$ is due to the resolution of normal force and one $\cos\alpha$ is due to the resolution of the slicing plane on which the force is acting. Due to trigonometric identities (Eq. (2.30)), the double resolution of stresses is hidden in the term $\cos 2\alpha$ of the Mohr circle equation (Eq. (2.31)).

Each point P on the Mohr circle (Fig. 2.7a) states values of normal and shear stress on the arbitrary plane within the body. For a given set of principal stresses (reference frame σ_1, σ_2), we can compute a second set of normal and shear stresses (frame σ, τ) at arbitrary (angle α) oriented surfaces. The values of normal and shear stress versus angle is seen in Fig. 2.7b for $\sigma_1 = 1$ MPa and $\sigma_2 = 0.5$ MPa. The transformation $[\sigma_1, \sigma_2] \rightarrow [\sigma, \tau]$ of stress components in 2D is analogous to the eigenvalue problem of the stress matrix in 3D visualized in Fig. 2.4. The points where the Mohr circle intersects the σ -axis represent principal planes. The associated σ -values are the principal stresses, σ_1 and σ_2 . The Mohr circle shows that the principal stresses are the maximum and minimum values of normal stresses in a body. The points

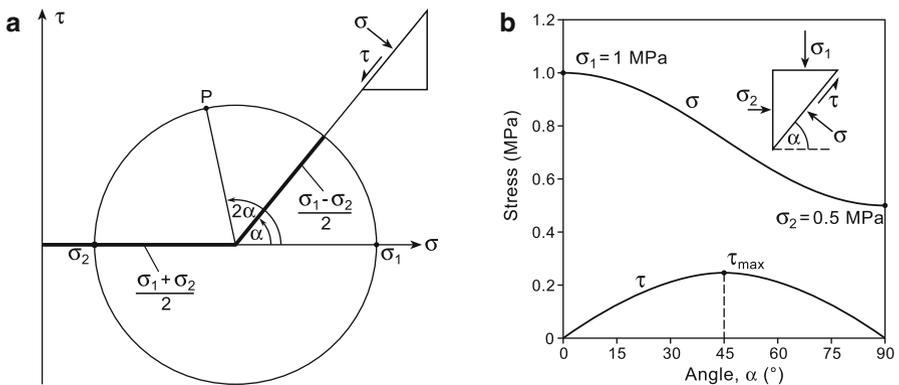


Fig. 2.7 Mohr circle of stress **a** in *stress space*, and **b** in *physical space*. The Mohr circle relates principal stresses (σ_1, σ_2) to normal and shear stresses on an arbitrary tilted plane (σ, τ, α) inside a deformable body

representing principal planes lie opposite the diameter in stress space. In physical space (Fig. 2.6, small prism), the planes are perpendicular since twice the rotation takes place on the Mohr circle. The *maximum shear stress* τ_{\max} occurs when $2\alpha = 90^\circ$. Thus the plane of maximum shear stress is oriented at 45° to the principal planes. Note that in rock mechanics notation, the positive τ -axis is upside down and positive shear stresses plot below the σ -axis. Also in times of fast personal computers, the Mohr circle of stress does not lose its significance for displaying stress, in particular with respect to the presentation of rock failure criteria (Chap. 3).

The Mohr circle is a way of plotting the normal and shear components for traction vectors associated with all possible planes through point P . Note the difference between the stress tensor at a point (Sect. 2.1) and the traction vector acting on some given plane through that point (Sect. 2.3). In *Mohr space* (= *stress space*), the normal and shear stress components of the stress vector with respect to the given plane are displayed. In *physical space*, the fracture plane within a rock mass after failure is typically inspected (Chap. 3). The orientation of the fracture plane is also governed by the internal friction of the material and therefore differs from the stress space.

Note-Box The Mohr circle of stress is obtained from balancing forces on a small prism within a deformable body under applied principal stresses. The Mohr circle relates normal and shear stress acting on an arbitrary slicing surface element of the prism to the applied principal stresses. The Mohr circle is a graphical method of transforming the stress tensor, and one way to visualize the stress field of a deformable body.

2.4 Visualizing Stress

In order to completely specify the state of stress in 2D (3D) with three (six) pieces of information, it is necessary to know the values of the stress components at each point on the body or, alternatively, to know the two (three) principal stresses and one (three) principal stress direction(s). Although it is difficult to display all of these data, there are a number of graphical methods that are useful in giving a partial picture of the stress field. As we know from Chap. 1, a stress field describes the way that the state of stress varies through space in a body. Since stress is not a straightforward descriptive quantity (Sect. 1.1, fictitious term; Sect. 2.1, abstract concept), we require appropriate techniques to display stress components and stress orientations.

One way to visualize stress magnitudes in 2D is *Lame's stress ellipse* combining Cartesian components of the stress vector (σ_x, σ_y) with components of principal stresses (σ_1, σ_2). Balancing forces according to the small prism presented in Fig. 2.8a it follows that

$$\left(\frac{\sigma_x}{\sigma_1}\right)^2 + \left(\frac{\sigma_y}{\sigma_2}\right)^2 = 1. \quad (2.32)$$

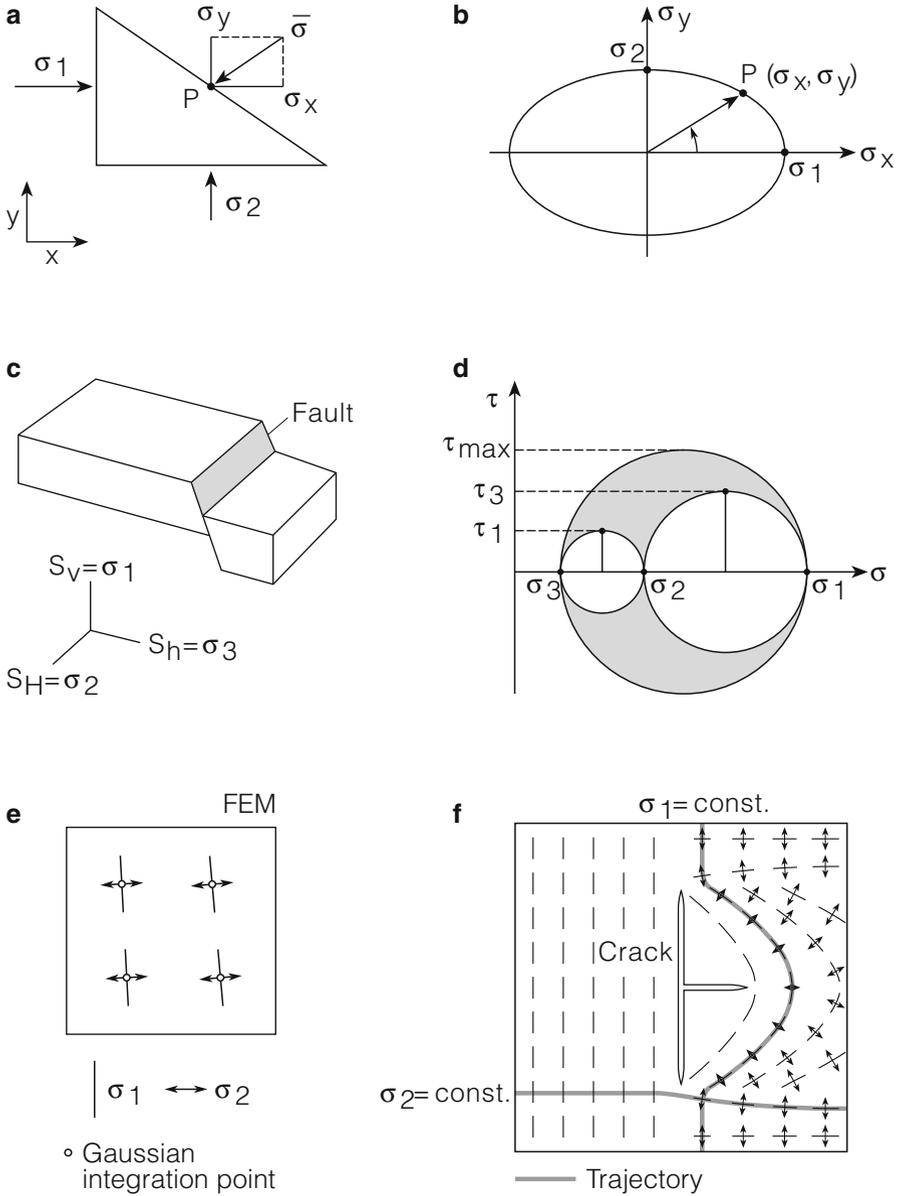


Fig. 2.8 Visualizing stress in physical space (*left*) and stress space (*right*). The relationship in 2D between a Cartesian frame and principal stresses (**a**) is shown by a stress ellipse (**b**). The relationship in 3D between a Mohr frame (normal and shear stress) and principal stresses of a normal fault in the Earth's crust (**c**) is shown by a Mohr circle of stress (**d**). Principal stress directions at points (**e**) are shown as stress trajectories (**f**)

Each vector from the origin to a point on the ellipse (Fig. 2.8b) represents a traction vector that acts on some plane passing through the point at which the principal stresses are (σ_1, σ_2) . However, although Lamé's stress ellipse shows the various traction vectors that act on different planes, it does not indicate the plane on which the given traction acts. It can be shown that in 3D the locus of all points traced out by the stress vector for all possible orientations of the plane on which it is acting defines an stress ellipsoid, the equation of which is given, e.g., in Jaeger and Cook (1979). The *stress ellipsoid* is one way to provide a 3D description of the state of stress at a single point in a deformable body. Note that the concept of stress can also be associated with that of a continuum and therefore is of value only at a scale at which the continuum concept is valid. The minimum volume for which an equivalent volume can be defined is termed the representative elementary volume (Chap. 10, REV). Generally, the continuum concept is of interest when the volume under investigation is at least two orders of magnitude larger than that of the REV (Hudson et al. 2003). In conclusion, a finite REV is needed for the physical definition of the concept of stress, while a point (zero volume) in Eq. (2.1) is needed for the mathematical definition of the concept of stress.

Our second way of visualizing stress components is the Mohr circle but, instead of 2D (Sect. 2.3), we now consider a 3D state of stress determined by three principal stresses and their orientations. Using Cauchy's second law, the stress vector components can be written

$$\begin{aligned}\sigma^2 + \tau^2 &= \sigma_i \sigma_i = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \\ \sigma &= \sigma_i n_i = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\end{aligned}\quad (2.33)$$

With the help of $n_i n_i = 1$, e.g., the identity

$$\left(\sigma - \frac{\sigma_2 - \sigma_3}{2}\right)^2 + \tau^2 = -\sigma(\sigma_2 + \sigma_3) + \left(\frac{\sigma_2 + \sigma_3}{2}\right)^2 + (\sigma^2 + \tau^2), \quad (2.34)$$

we obtain the expression

$$\left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 + \tau^2 = n_1^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2. \quad (2.35)$$

Mathematically, this is the equation of a circle with its centre at $(\sigma = (\sigma_2 + \sigma_3)/2, \tau = 0)$ and a radius which depends on n_1 . Since $0 \leq n_1^2 \leq 1$, the minimum centre point distance of stress points is $(\sigma_2 - \sigma_3)/2 = \tau_1$ for $n_1 = 0$, whereas the maximum distance is $\sigma_1 + (\sigma_2 - \sigma_3)/2$ for $n_1 = \pm 1$. Analogue findings on two additional equations following from (2.35) by cyclic permutation of indices lead to the *Mohr circles of stress in 3D*. Arranging principal stresses with respect to magnitude ($\sigma_1 \geq \sigma_2 \geq \sigma_3$), we obtain the stress points within the shaded area of Fig. 2.8d. The circles with radii τ_i correspond to slicing planes with the surface normal perpendicular to one of the three principal axes.

Our third way of visualizing stress is restricted to the direction of principal stresses. Like in *Finite Element* (FE) codes where principal stress directions are displayed at Gaussian integration points within a finite element (Fig. 2.8e, dots), we can define stress trajectories. A line whose tangent at every point is in the direction of a principal stress component is called a *stress trajectory*. In the 2D rock model with a T-shaped crack (Fig. 2.8f), two stress trajectories are shown, one for the direction of the largest and the other for the direction of the least principal stress. As principal stresses are always at right angles to each other, stress trajectories form an orthogonal set of lines (Fig. 2.8f, σ_1 perpendicular σ_2). In Exercise 2.2, the reader can complete a dense network of stress trajectories for the model sketched in Fig. 2.8f using stresses at all Gaussian integration points within the full finite element model.

Analogue to the lines of electric (or magnetic) flux where electric (or magnetic) fields are visualized, stress trajectories are used to visualize the elastic stress field in a deformable body. An early picture of electric flux lines is shown in Fig. 2.9, where the Russian Jakob von Narkievicz-Jodko electrified a human hand in 1895 and saved the picture on a photographic plate. A bunch of electric lines was captured in silver gelatine to give an idea of the complex internal electric field of a human hand. You can imagine that also in the case of a complex rock body with or



Fig. 2.9 In 1895, Jakob von Narkievicz-Jodko electrified a human hand in order to capture the electric flux line network in silver gelatine

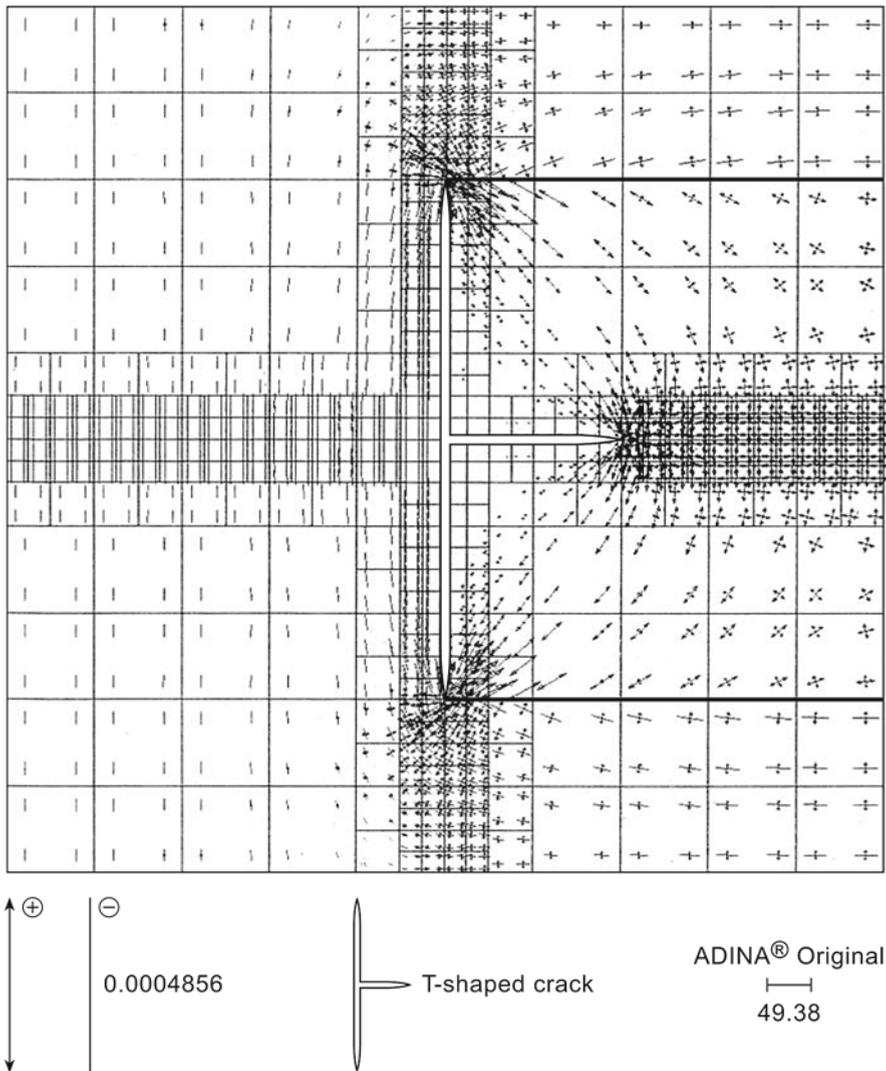


Fig. 2.10 Finite element mesh of a T-shaped crack formed during cooling of a 2D periodic rock model showing directions and magnitudes of σ_1 and σ_2 at every single Gaussian integration point within each of the 436 quadratic finite elements (Zang 1991)

without excavation surfaces, many data points of principal stresses are needed to draw a realistic network of stress trajectories characterizing the elastic stress field (see Exercise 2.2).

Among the experimental techniques to visualize stress are *isochromatics* (constant maximum shear stress, see Sect. 6.4), *isopachs* (curves along which the mean normal stress is constant, e.g. Durelli et al. (1958) electrical conducting paper), *isoclines* (curve on which principal axes make a constant angle with a given fixed

reference direction, photoelasticity), and fault slip lines (curves on which the shear stress is maximum, structural geology).

Exercise 2.2 Stress Trajectories. Consider the finite element mesh with a T-shaped crack in Fig. 2.10. Put a transparency foil on top of the figure.

- (a) Draw a set of solid lines onto the foil following the σ_1 -orientation in the rock model using the stress orientations given at Gaussian integration points. Add a second set of lines with a different color, which follows the σ_2 -orientation.
- (b) What may be the reason for the rotation of the stress field in the rock model near the two marked bold solid lines?

Note that at crack tips, stress magnitudes increase (Fig. 2.10, mesh refinement) and stress orientations may change quickly over short distances. The reason for this will be expanded in Chap. 3, where crack-tip stress singularities are treated. As is true for large rock mass excavation surfaces, also small cracks in rock serve as free surfaces and therefore are principal normal stress planes.

Note-Box The way that the state of stress varies through space (stress field) can be visualized, e.g., by firstly a stress ellipse (ellipsoid), secondly a Mohr stress circle, and thirdly stress trajectories. Along a stress trajectory (line), the direction of principal stress is tangential in every point. In the first two methods, the state of stress is described at a point of a deformable body. Lamé's stress ellipse defines the locus of all points traced out by the stress vector for all possible orientations of the plane on which it is acting. The normal and shear stress components of the stress vector with respect to the given plane are visualized in the Mohr circle.



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