Chapter 2

SUPERQUADRICS AND THEIR GEOMETRIC PROPERTIES

In this chapter we define superquadrics after we outline a brief history of their development. Besides giving basic superquadric equations, we derive also some other useful geometric properties of superquadrics.

2.1 SUPERELLIPSE

A superellipse is a closed curve defined by the following simple equation

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1, \tag{2.1}$$

where a and b are the size (positive real number) of the major and minor axes and m is a rational number

$$m = \frac{p}{q} > 0, \quad \text{where} \begin{cases} p & \text{is an even positive integer,} \\ q & \text{is an odd positive integer.} \end{cases} \tag{2.2}$$

If $m = 2$ and $a = b$, we get the equation of a circle. For larger m, however, we gradually get more rectangular shapes, until for $m \to \infty$ the curve takes up a rectangular shape (Fig. 2.1). On the other hand, when $m \to 0$ the curve takes up the shape of a cross.

Superellipses are special cases of curves which are known in analytical geometry as Lamé curves, where m can be any rational number (Loria, 1910). Lamé curves are named after the French mathematician Gabriel Lamé, who was the first who described these curves in the early 19th century[1].

[1] *Gabriel Lamé. Examen des différentes méthodes employées pour résoudre les problémes de geometrie, Paris, 1818.*

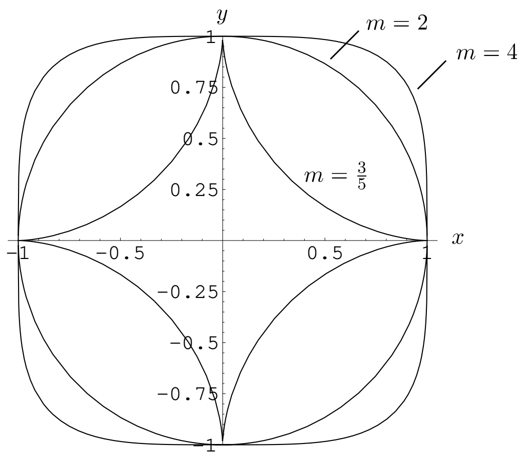


Figure 2.1. A superellipse can change continuously from a star-shape through a circle to a square shape in the limit ($m \to \infty$).

Piet Hein, a Danish scientist, writer and inventor, popularized these curves for design purposes in the 1960s (Gardner, 1965). Faced with various design problems Piet Hein proposed a shape that mediates between circular and rectangular shapes and named it a *superellipse*. Piet Hein designed the streets and an underground shopping area on Sergels Torg in Stockholm in the shape of concentric superellipses with $m = 2.5$. Other designers used superellipse shapes for design of table tops and other furniture. Piet Hein also made a generalization of superellipse to 3D which he named *superellipsoids* or *superspheres*. He named superspheres with $m = 2.5$ and the height-width ratio of 4:3 *supereggs* (Fig. 2.2). Though it looks as if a superegg standing on either of its ends should topple over, it does not because the center of gravity is lower than the center of curvature! According to Piet Hein this spooky stability of the superegg can be taken as symbolic of the superelliptical balance between the orthogonal and the round.

Superellipses were used for lofting in the preliminary design of aircraft fuselage (Flanagan and Hefner, 1967; Faux and Pratt, 1985). In 1981, Barr generalized the superellipsoids to a family of 3D shapes that he named *superquadrics* (Barr, 1981). He introduced the notation common in the computer vision literature and also used in this book. Barr saw the importance of superquadric models in particular for computer graphics and for three-dimensional design since superquadric models, which compactly represent a continuum of useful forms with rounded edges, can easily be rendered and shaded and further deformed by parametric deformations.

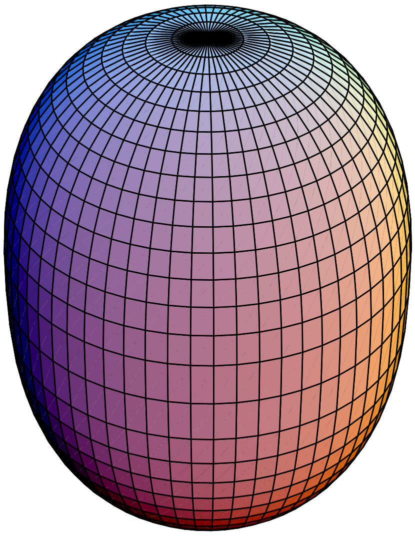


Figure 2.2. A "superegg" (superquadric with $m = 2.5$, height-width ratio = 4:3) is stable in the upright position because the center of gravity is lower than the center of curvature (Gardner, 1965).

2.1.1 LAMÉ CURVES

For Lamé curves

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1, \tag{2.3}$$

m can be any rational number. From the topological point of view, there are nine different types of Lamé curves depending on the form of the exponent m in equation (2.3) which is defined by positive integers $k, h \in N$ (Loria, 1910).

Lamé curves with positive m are

1. $m = \frac{2h}{2k+1} > 1$ (Fig. 2.3 a)
2. $m = \frac{2h}{2k+1} < 1$ (Fig. 2.3 b)
3. $m = \frac{2h+1}{2k} > 1$ (Fig. 2.3 c)
4. $m = \frac{2h+1}{2k} < 1$ (Fig. 2.3 d)
5. $m = \frac{2h+1}{2k+1} > 1$ (Fig. 2.3 e)
6. $m = \frac{2h+1}{2k+1} < 1$ (Fig. 2.3 f)

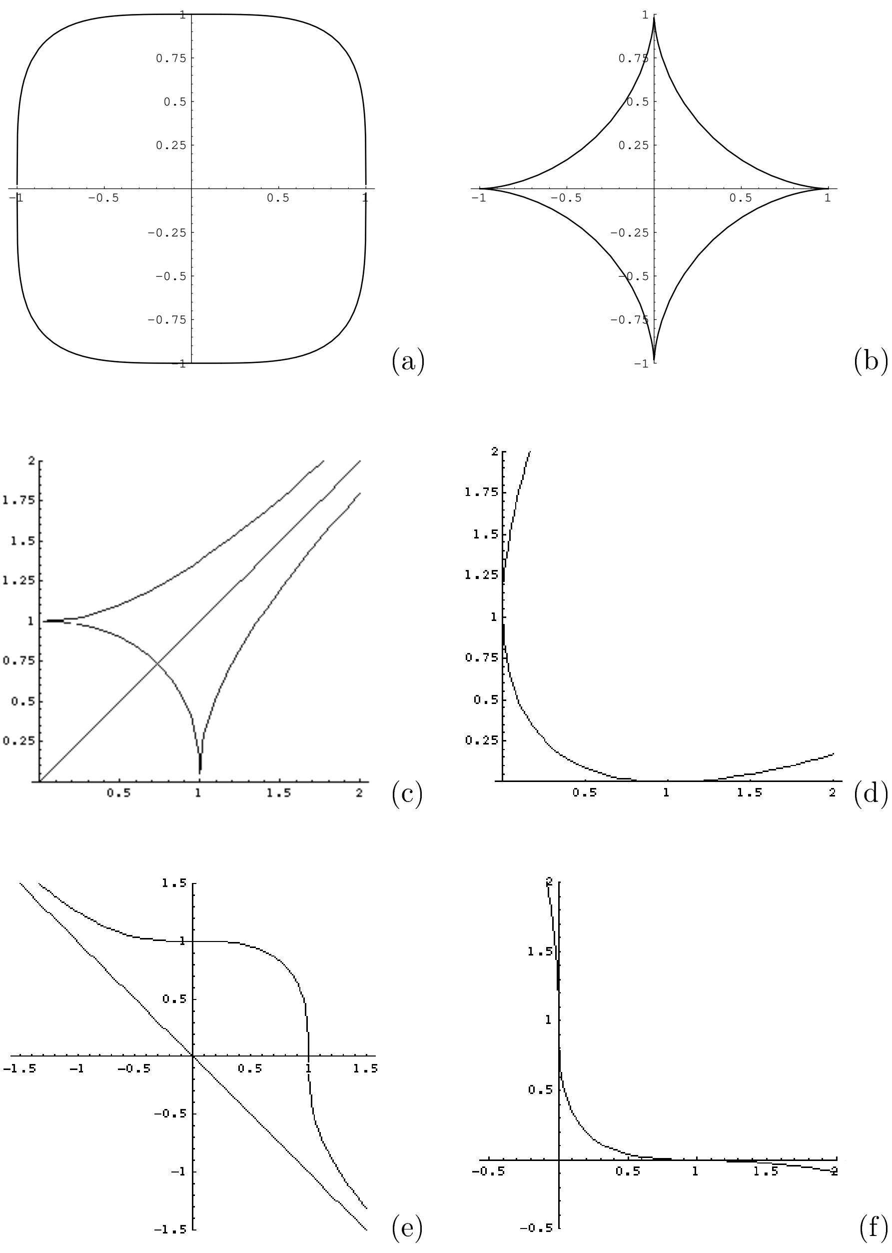
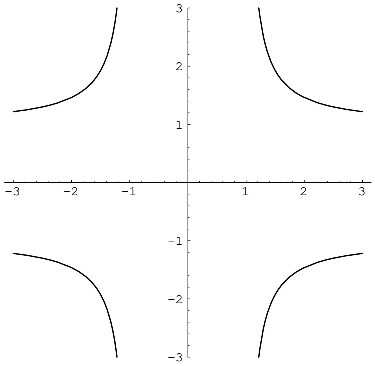
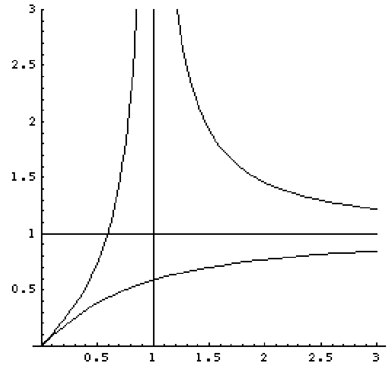


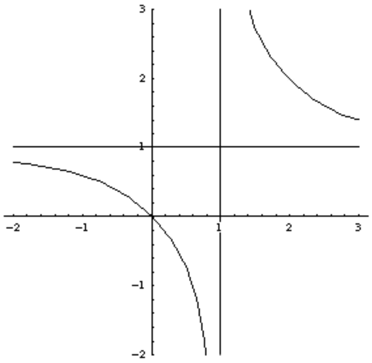
Figure 2.3. Lamé curves with positive m. Only the first two types (a) and (b) are superellipses.



(g)



(h)



(i)

Figure 2.3 (continued). Lamé curves with negative m

Lamé curves with negative m are

7. $m = -\frac{2h}{2k+1}$ (Fig. 2.3 g)

8. $m = -\frac{2h+1}{2k}$ (Fig. 2.3 h)

9. $m = -\frac{2h+1}{2k+1}$ (Fig. 2.3 i)

which are shown in Fig. 2.3. Only the first type of Lamé curves (Fig. 2.3 a) are superellipses in the strict sense, but usually also the second type (Fig. 2.3 b) is included since the only difference is in the value of the exponent m (< 1 or > 1). Superellipses can therefore be written as

$$\left(\frac{x}{a}\right)^{\frac{2}{\varepsilon}} + \left(\frac{y}{b}\right)^{\frac{2}{\varepsilon}} = 1, \tag{2.4}$$

where ε can be any positive real number if the two terms are first raised to the second power.

2.2 SUPERELLIPSOIDS AND SUPERQUADRICS

A 3D surface can be obtained by the spherical product of two 2D curves (Barr, 1981). A unit sphere, for example, is produced when a half circle in a plane orthogonal to the (x, y) plane (Fig. 2.4)

$$\mathbf{m}(\eta) = \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}, \qquad -\pi/2 \leq \eta \leq \pi/2 \tag{2.5}$$

is crossed with the full circle in (x, y) plane

$$\mathbf{h}(\omega) = \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}, \qquad -\pi \leq \omega < \pi, \tag{2.6}$$

$$\mathbf{r}(\eta, \omega) = \mathbf{m}(\eta) \otimes \mathbf{h}(\omega) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\eta\cos\omega \\ \cos\eta\sin\omega \\ \sin\eta \end{bmatrix}, \qquad \begin{matrix} -\pi/2 \leq \eta \leq \pi/2 \\ -\pi \leq \omega < \pi \end{matrix}. \tag{2.7}$$

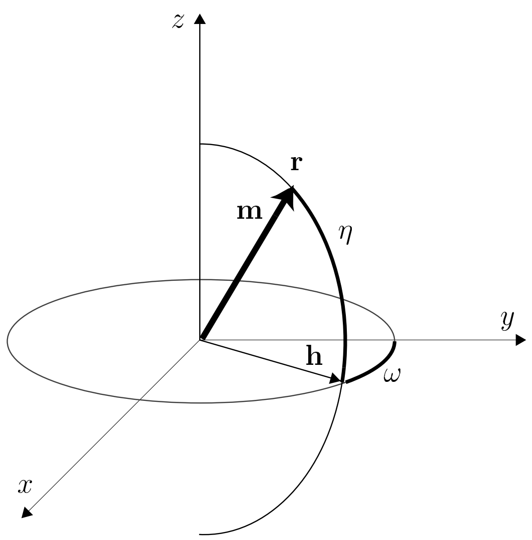


Figure 2.4. A 3D vector **r**, which defines a closed 3D surface, can be obtained by a spherical product of two 2D curves.

Analogous to a circle, a superellipse

$$\left(\frac{x}{a}\right)^{\frac{2}{\varepsilon}} + \left(\frac{y}{b}\right)^{\frac{2}{\varepsilon}} = 1 \tag{2.8}$$

can be written as

$$\mathbf{s}(\theta) = \begin{bmatrix} a\cos^{\varepsilon}\theta \\ b\sin^{\varepsilon}\theta \end{bmatrix}, \qquad -\pi \le \theta \le \pi \quad . \tag{2.9}$$

Note that exponentiation with ε is a *signed power function* such that $\cos^{\varepsilon}\theta = \text{sign}(\cos\theta)|\cos\theta|^{\varepsilon}$! Superellipsoids can therefore be obtained by a spherical product of a pair of such superellipses

$$\begin{aligned} \mathbf{r}(\eta,\omega) &= \mathbf{s}_1(\eta)\otimes\mathbf{s}_2(\omega) = \qquad (2.10)\\ &= \begin{bmatrix} \cos^{\varepsilon_1}\eta \\ a_3\sin^{\varepsilon_1}\eta \end{bmatrix} \otimes \begin{bmatrix} a_1\cos^{\varepsilon_2}\omega \\ a_2\sin^{\varepsilon_2}\omega \end{bmatrix} = \\ &= \begin{bmatrix} a_1\cos^{\varepsilon_1}\eta\cos^{\varepsilon_2}\omega \\ a_2\cos^{\varepsilon_1}\eta\sin^{\varepsilon_2}\omega \\ a_3\sin^{\varepsilon_1}\eta \end{bmatrix}, \qquad \begin{matrix} -\pi/2 \le \eta \le \pi/2 \\ -\pi \le \omega < \pi \end{matrix} \quad . \end{aligned}$$

Parameters a_1, a_2 and a_3 are scaling factors along the three coordinate axes. ε_1 and ε_2 are derived from the exponents of the two original superellipses. ε_2 determines the shape of the superellipsoid cross section parallel to the (x, y) plane, while ε_1 determines the shape of the superellipsoid cross section in a plane perpendicular to the (x, y) plane and containing z axis (Fig. 2.5).

An alternative, implicit superellipsoid equation can be derived from the explicit equation using the equality $\cos^2\alpha + \sin^2\alpha = 1$. We rewrite equation (2.10) as follows:

$$\left(\frac{x}{a_1}\right)^2 = \cos^{2\varepsilon_1}\eta\ \cos^{2\varepsilon_2}\omega \ , \tag{2.11}$$

$$\left(\frac{y}{a_2}\right)^2 = \cos^{2\varepsilon_1}\eta\ \sin^{2\varepsilon_2}\omega \ , \tag{2.12}$$

$$\left(\frac{z}{a_3}\right)^2 = \sin^{2\varepsilon_1}\eta \ . \tag{2.13}$$

Raising both sides of equations (2.11) and (2.12) to the power of $1/\varepsilon_2$ and then adding respective sides of these two equations gives

$$\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}} = \cos^{\frac{2\varepsilon_1}{\varepsilon_2}}\eta \ . \tag{2.14}$$

Next, we raise both sides of equation (2.13) to the power of $1/\varepsilon_1$ and both sides of equation (2.14) to the power of $\varepsilon_2/\varepsilon_1$. By adding the respective

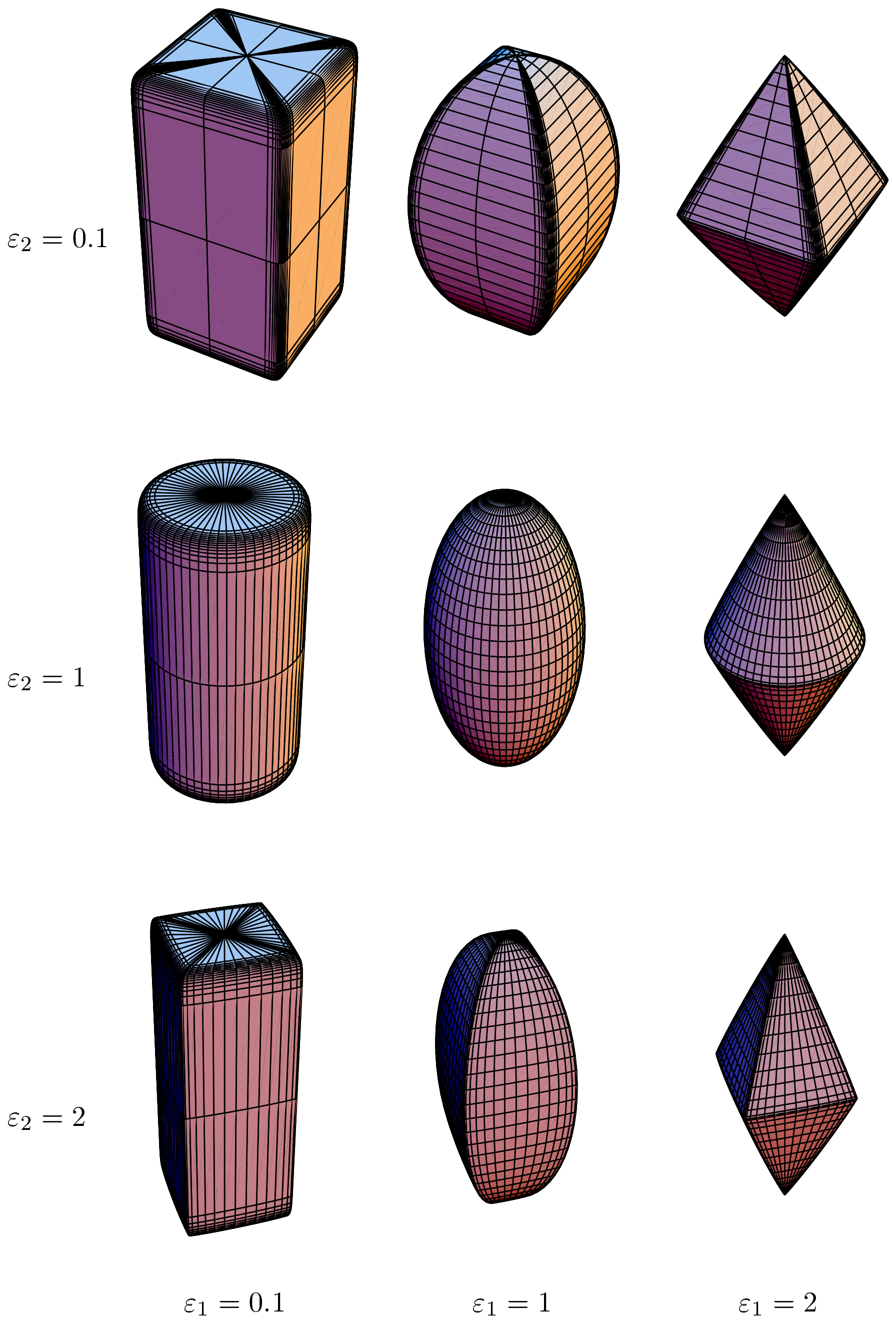


Figure 2.5. Superellipsoids with different values of exponents ε_1 and ε_2. Size parameters a_1, a_2, a_3 are kept constant. Superquadric-centered coordinate axis z points upwards!

sides of these two equations we get the implicit superquadric equation

$$\left(\left(\frac{x}{a_1} \right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2} \right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{\varepsilon_1}} + \left(\frac{z}{a_3} \right)^{\frac{2}{\varepsilon_1}} = 1. \tag{2.15}$$

All points with coordinates (x, y, z) that correspond to the above equation lie, by definition, on the surface of the superellipsoid.

The function

$$F(x, y, z) = \left(\left(\frac{x}{a_1} \right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2} \right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{\varepsilon_1}} + \left(\frac{z}{a_3} \right)^{\frac{2}{\varepsilon_1}} \tag{2.16}$$

is also called the *inside-outside* function because it provides a simple test whether a given point lies inside or outside the superquadric. If $F < 1$, the given point (x, y, z) is inside the superquadric, if $F = 1$ the corresponding point lies on the surface of the superquadric, and if $F > 1$ the point lies outside the superquadric.

A special case of superellipsoids when $\varepsilon_1 = \varepsilon_2$

$$\left(\frac{x}{a} \right)^{2m} + \left(\frac{y}{b} \right)^{2m} + \left(\frac{z}{c} \right)^{2m} = 1 \tag{2.17}$$

was already studied by S. Spitzer[2]. Spitzer computed the area of superellipse and the volume of this special superellipsoid when m is a natural number (Loria, 1910).

2.2.1 SUPERQUADRICS

The term *superquadrics* was defined by Barr in his seminal paper (Barr, 1981). Superquadrics are a family of shapes that includes not only superellipsoids, but also superhyperboloids of one piece and superhyperboloids of two pieces, as well as supertoroids (Fig. 2.6). In computer vision literature, it is common to refer to superellipsoids by the more generic term of superquadrics. In this book we also use the term superquadrics as a synonym for superellipsoids.

By means of introducing parametric exponents of trigonometric functions, Barr made a generalization not only of ellipsoids, but also of the other two standard quadric surfaces; hyperboloids of one sheet

$$\left(\frac{x}{a_1} \right)^2 + \left(\frac{y}{a_2} \right)^2 - \left(\frac{z}{a_3} \right)^2 = 1 \tag{2.18}$$

[2] *Arch. Math. Phys. LXI, 1877.*

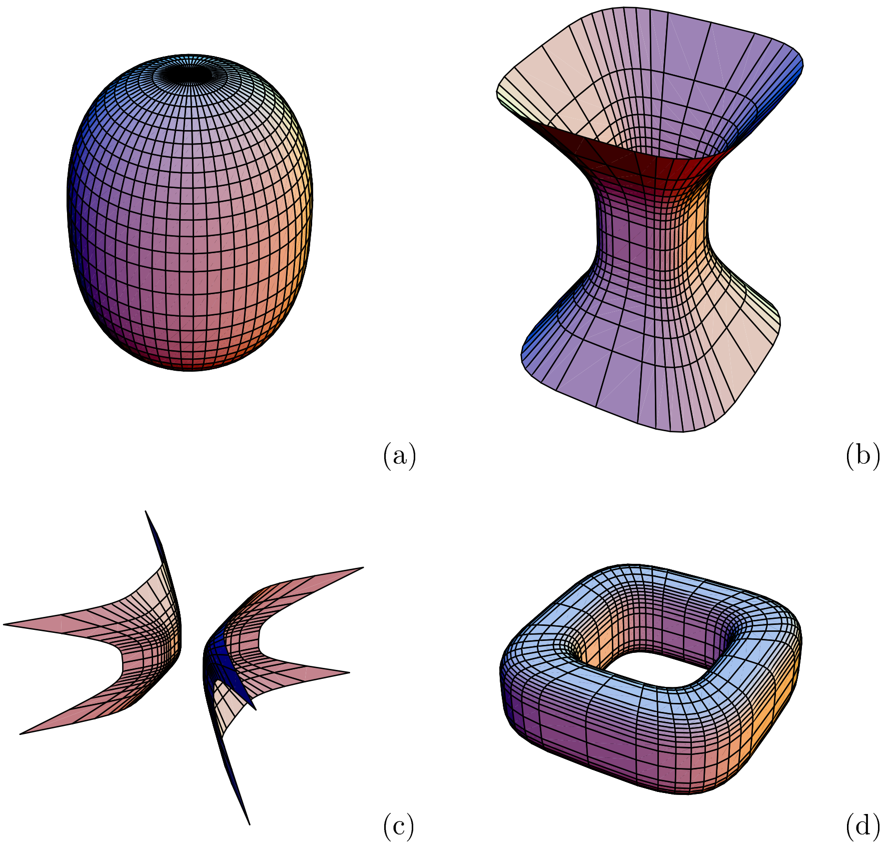


Figure 2.6. Superquadrics are a family of shapes that includes (a) superellipsoids, (b) superhyperboloids of one, and (c) of two pieces, and (d) supertoroids.

and hyperboloids of two sheets

$$\left(\frac{x}{a_1}\right)^2 - \left(\frac{y}{a_2}\right)^2 - \left(\frac{z}{a_3}\right)^2 = 1 \quad . \tag{2.19}$$

Superhyperboloids of one piece are therefore defined by the surface vector

$$\begin{aligned} \mathbf{r}(\eta, \omega) &= \begin{bmatrix} \sec^{\varepsilon_1} \eta \\ a_3 \tan^{\varepsilon_1} \eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \cos^{\varepsilon_2} \omega \\ a_2 \sin^{\varepsilon_2} \omega \end{bmatrix} = \\ &= \begin{bmatrix} a_1 \sec^{\varepsilon_1} \eta \cos^{\varepsilon_2} \omega \\ a_2 \sec^{\varepsilon_1} \eta \sin^{\varepsilon_2} \omega \\ a_3 \tan^{\varepsilon_1} \eta \end{bmatrix}, \qquad \begin{matrix} -\pi/2 < \eta < \pi/2 \\ -\pi \leq \omega < \pi \end{matrix} \end{aligned} \tag{2.20}$$

and by the implicit function

$$F(x,y,z) = \left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{\varepsilon_1}} - \left(\frac{z}{a_3}\right)^{\frac{2}{\varepsilon_1}} \quad . \tag{2.21}$$

Superhyperboloids of two pieces are defined by the surface vector

$$\begin{aligned} \mathbf{r}(\eta,\omega) &= \begin{bmatrix} \sec^{\varepsilon_1}\eta \\ a_3 \tan^{\varepsilon_1}\eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \sec^{\varepsilon_2}\omega \\ a_2 \tan^{\varepsilon_2}\omega \end{bmatrix} = \\ &= \begin{bmatrix} a_1 \sec^{\varepsilon_1}\eta \sec^{\varepsilon_2}\omega \\ a_2 \sec^{\varepsilon_1}\eta \tan^{\varepsilon_2}\omega \\ a_3 \tan^{\varepsilon_1}\eta \end{bmatrix}, \quad \begin{array}{l} -\pi/2 < \eta < \pi/2 \\ -\pi/2 < \omega < \pi/2 \ \text{(sheet 1)} \\ \pi/2 < \omega < 3\pi/2 \ \text{(sheet 2)} \end{array} \end{aligned} \tag{2.22}$$

and by the implicit function

$$F(x,y,z) = \left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} - \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{\varepsilon_1}} - \left(\frac{z}{a_3}\right)^{\frac{2}{\varepsilon_1}} \quad . \tag{2.23}$$

A torus is a special case of extended quadric surface

$$(r-a)^2 = \left(\frac{z}{a_3}\right)^2 = 1 \quad , \tag{2.24}$$

where

$$r = \sqrt{\left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{a_2}\right)^2} \quad . \tag{2.25}$$

Supertoroids are therefore defined by the following surface vector

$$\begin{aligned} \mathbf{r}(\eta,\omega) &= \begin{bmatrix} a_4 + \cos^{\varepsilon_1}\eta \\ a_3 \sin^{\varepsilon_1}\eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \cos^{\varepsilon_2}\omega \\ a_2 \sin^{\varepsilon_2}\omega \end{bmatrix} = \\ &= \begin{bmatrix} a_1(a_4 + \cos^{\varepsilon_1}\eta)\cos^{\varepsilon_2}\omega \\ a_2(a_4 + \cos^{\varepsilon_1}\eta)\sin^{\varepsilon_2}\omega \\ a_3 \sin^{\varepsilon_1}\eta \end{bmatrix}, \quad \begin{array}{l} -\pi \le \eta < \pi \\ -\pi \le \omega < \pi \end{array} \end{aligned} \tag{2.26}$$

and by the implicit function

$$F(x,y,z) = \left(\left(\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{2}} - a_4 \right)^{\frac{2}{\varepsilon_1}} + \left(\frac{z}{a_3}\right)^{\frac{2}{\varepsilon_1}} \quad , \tag{2.27}$$

where a_4 is a positive real offset value which is related to the radius of the supertoroid in the following way

$$a_4 = \frac{R}{\sqrt{a_1^2 + a_2^2}} \quad . \tag{2.28}$$

2.3 SUPERQUADRICS IN GENERAL POSITION

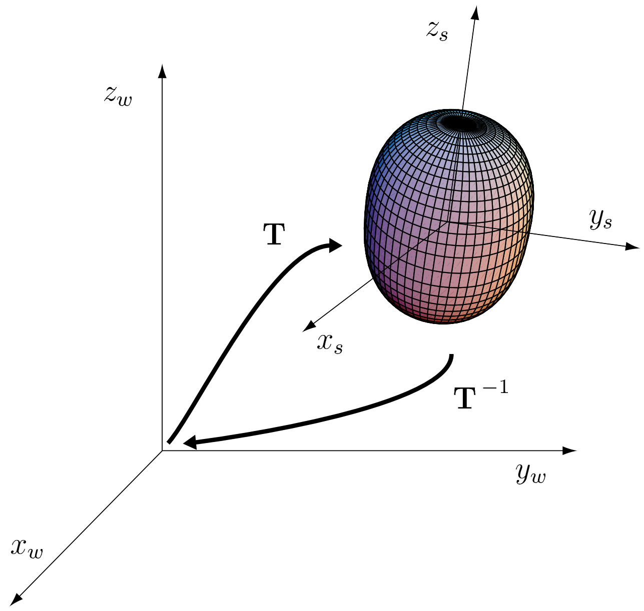


Figure 2.7. To define a superquadric in general position six additional parameters are needed.

A superellipsoid in the local or superquadric centered coordinate system (x_s, y_s, z_s) is defined by 5 parameters (3 for size in each dimension and 2 for shape defining exponents). To model or to recover superellipsoids or superquadrics from image data we must represent superquadrics in general position or in a global coordinate system. A superquadric in general position requires 6 additional parameters for expressing the rotation and translation of the superquadric relative to the center of the world coordinate system (x_w, y_w, z_w). One can use different conventions to define translation and rotation. We use a homogeneous coordinate transformation $\mathbf{T}$ to transform the 3D points expressed in the superquadric centered coordinate system $[x_s, y_s, z_s, 1]^T$ into the world coordinates $[x_w, y_w, z_w, 1]^T$ (Fig. 2.7)

$$\begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix} , \qquad (2.29)$$

where

$$\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.30}$$

For a given point, transformation $\mathbf{T}$ first rotates (defined by the parameters n, o and a) that point and then translates it for $[p_x, p_y, p_z, 1]^T$ (Paul, 1981). Since we need in our equations the points to be expressed in superquadric centered coordinates, we have to compute them from the world coordinates

$$\begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix} = \mathbf{T}^{-1} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} . \tag{2.31}$$

Transformation $\mathbf{T^{-1}}$ performs the two operations in reverse order—it first translates a point and then rotates it.

Inverting homogeneous transformation matrix $\mathbf{T}$ gives

$$\mathbf{T^{-1}} = \begin{bmatrix} n_x & n_y & n_z & -(p_x n_x + p_y n_y + p_z n_z) \\ o_x & o_y & o_z & -(p_x o_x + p_y o_y + p_z o_z) \\ a_x & a_y & a_z & -(p_x a_x + p_y a_y + p_z a_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.32}$$

By substituting equations (2.29) and (2.32) into equation (2.15), we get the inside-outside function for superquadrics in general position and orientation

$$\begin{aligned} F(x_w, y_w, z_w) = & \\ & \left(\left(\frac{n_x x_w + n_y y_w + n_z z_w - p_x n_x - p_y n_y - p_z n_z}{a_1} \right)^{\frac{2}{\varepsilon_2}} + \right. \\ & \left. + \left(\frac{o_x x_w + o_y y_w + o_z z_w - p_x o_x - p_y o_y - p_z o_z}{a_2} \right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{\varepsilon_1}} + \\ & + \left(\frac{a_x x_w + a_y y_w + a_z z_w - p_x a_x - p_y a_y - p_z a_z}{a_3} \right)^{\frac{2}{\varepsilon_1}} . \end{aligned} \tag{2.33}$$

We use Euler angles (ϕ, θ, ψ) to express the elements of the rotational part of transformation matrix $\mathbf{T}$. Euler angles define orientation in terms of rotation ϕ about the z axis, followed by a rotation θ about the new y axis, and finally, a rotation ψ about the new z axis (Paul, 1981)

$$\mathbf{T} = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta & p_x \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\theta & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\theta & \sin\phi\sin\theta & p_y \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.34}$$

The inside-outside function for superquadrics in general position has therefore, 11 parameters

$$F(x_w, y_w, z_w) = F(x_w, y_w, z_w; a_1, a_2, a_3, \varepsilon_1, \varepsilon_2, \phi, \theta, \psi, p_x, p_y, p_z), \tag{2.35}$$

where a_1, a_2, a_3 define the superquadric size; ε_1 and ε_2 the shape; ϕ, θ, ψ the orientation, and p_x, p_y, p_z the position in space. We refer to the set of all model parameters as $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{11}\}$.

2.4 SOME GEOMETRIC PROPERTIES OF SUPERELLIPSOIDS

In this section, derivations of superellipsoid normal vector, radial Euclidean distance between a point and a superellipsoid, rim of a superellipsoid in general orientation, area and inertial moments of superellipse, as well as volume and inertial moments of superellipsoids are given. Zarrugh (Zarrugh, 1985) proposed numerical methods for computing volume and moments of inertia for superellipsoids. Here we derive these properties analytically.

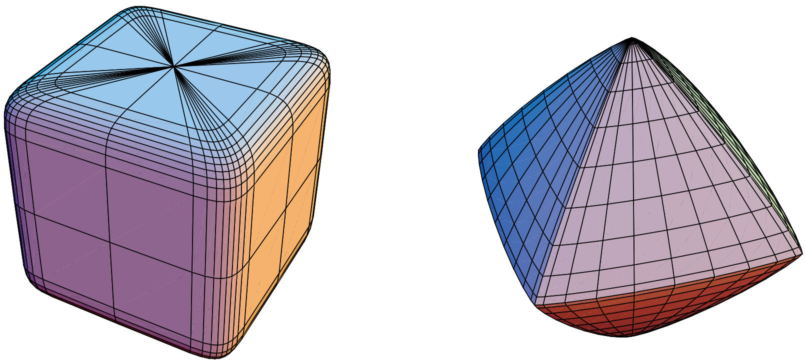
2.4.1 NORMAL VECTOR OF THE SUPERELLIPSOID SURFACE

Normal vector at a point $\mathbf{r}(\eta, \omega)$ on the superellipsoid surface (Eq. (2.10) on page 19) is defined by a cross product of the tangent vectors along the coordinate curves

$$\mathbf{n}(\eta, \omega) = \mathbf{r}_\eta(\eta, \omega) \times \mathbf{r}_\omega(\eta, \omega) =$$

$$= \begin{bmatrix} -a_1\varepsilon_1 \sin\eta \cos^{\varepsilon_1-1}\eta \cos^{\varepsilon_2}\omega \\ -a_2\varepsilon_1 \sin\eta \cos^{\varepsilon_1-1}\eta \sin^{\varepsilon_2}\omega \\ a_3\varepsilon_1 \sin^{\varepsilon_1-1}\eta \cos\eta \end{bmatrix} \times \begin{bmatrix} -a_1\varepsilon_2 \cos^{\varepsilon_1}\eta \sin\omega \cos^{\varepsilon_2-1}\omega \\ a_2\varepsilon_2 \cos^{\varepsilon_1}\eta \cos\omega \sin^{\varepsilon_2-1}\omega \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -a_2a_3\varepsilon_1\varepsilon_2 \sin^{\varepsilon_1-1}\eta \cos^{\varepsilon_1+1}\eta \cos\omega \sin^{\varepsilon_2-1}\omega \\ -a_1a_3\varepsilon_1\varepsilon_2 \sin^{\varepsilon_1-1}\eta \cos^{\varepsilon_1+1}\eta \sin\omega \cos^{\varepsilon_2-1}\omega \\ -a_1a_2\varepsilon_1\varepsilon_2 \sin\eta \cos^{2\varepsilon_1-1}\eta \sin^{\varepsilon_2-1}\omega \cos^{\varepsilon_2-1}\omega \end{bmatrix}. \tag{2.36}$$



(a) (b)

Figure 2.8. Every (a) superquadric has a (b) dual superquadric which is defined by the scaled normal vector of the original one.

The above expression can be simplified by defining the following common term

$$f(\eta,\omega) = -a_1 a_2 a_3 \varepsilon_1 \varepsilon_2 \sin^{\varepsilon_1 - 1}\eta \cos^{2\varepsilon_1 - 1}\eta \sin^{\varepsilon_2 - 1}\omega \cos^{\varepsilon_2 - 1}\omega \quad , \tag{2.37}$$

so that the normal vector can be written as

$$\mathbf{n}(\eta,\omega) = f(\eta,\omega) \begin{bmatrix} \frac{1}{a_1} \cos^{2-\varepsilon_1}\eta \cos^{2-\varepsilon_2}\omega \\ \frac{1}{a_2} \cos^{2-\varepsilon_1}\eta \sin^{2-\varepsilon_2}\omega \\ \frac{1}{a_3} \sin^{2-\varepsilon_1}\eta \end{bmatrix} \quad . \tag{2.38}$$

The scalar function $f(\eta,\omega)$ term can be dropped out if we only need the surface normal direction. By doing this, we actually get a dual superquadric to the original superquadric $\mathbf{r}(\eta,\omega)$ (Barr, 1981)

$$\mathbf{n}_d(\eta,\omega) = \begin{bmatrix} \frac{1}{a_1} \cos^{2-\varepsilon_1}\eta \cos^{2-\varepsilon_2}\omega \\ \frac{1}{a_2} \cos^{2-\varepsilon_1}\eta \sin^{2-\varepsilon_2}\omega \\ \frac{1}{a_3} \sin^{2-\varepsilon_1}\eta \end{bmatrix} \quad . \tag{2.39}$$

A superquadric and its dual superquadric are shown in Fig. 2.8. Using the explicit equation for superellipsoid surfaces (Eq. 2.10) we can express the normal vector also, in terms of the components of the surface vector

$$\mathbf{n}_d(\eta,\omega) = \begin{bmatrix} \frac{1}{x}\cos^2\eta\cos^2\omega \\ \\ \frac{1}{y}\cos^2\eta\sin^2\omega \\ \\ \frac{1}{z}\sin^2\eta \end{bmatrix} . \tag{2.40}$$

Function $\mathbf{n}_d(\eta,\omega)$ is derived from $\mathbf{r}(\eta,\omega)$ by replacing the parameters a_1, a_2, and a_3 with their reciprocal values, and the parameters ε_1 and ε_2 with their complementary values to the number 2. Now, if we interpret the vector function $\mathbf{n}_d(\eta,\omega)$ as a superellipsoid and construct the normal vector function as described above, we get the original superellipsoid $\mathbf{r}(\eta,\omega)$.

Note that those superquadrics which have ε_1 and $\varepsilon_2 > 2$ have sharp spiky corners where the normal vector is not uniquely defined.

2.4.2 DISTANCE BETWEEN A POINT AND A SUPERELLIPSOID

Although the true Euclidean distance between a point and a superellipsoid can be calculated by using numerical minimization, we do not know of any closed form solution in form of an algebraic expression. But for *radial Euclidean distance* such an expression can be derived based on the implicit superellipsoid equation (Whaite and Ferrie, 1991). The radial Euclidean distance is defined as a distance between a point and a superellipsoid along a line through the point and the center of a superellipsoid. We will summarize the derivation of this function and in Chapter 4 relate it to a distance measure that we proposed for recovery of superellipsoids from range data (Solina and Bajcsy, 1990).

The derivation is illustrated in Fig. 2.9. For a point defined by a vector $\mathbf{r_0} = (x_0, y_0, z_0)$ in the canonical coordinate system of a superellipsoid, we are looking for a scalar β, that scales the vector, so that the tip of the scaled vector $\mathbf{r_s} = \beta\mathbf{r_0}$ lies on the surface of the superellipsoid. Thus for the scaled vector $\mathbf{r_s}$, the following equation holds

$$F(\beta x_0, \beta y_0, \beta z_0) = \left[\left[\left(\frac{\beta x_0}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{\beta y_0}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right]^{\frac{\varepsilon_2}{\varepsilon_1}} + \left(\frac{\beta z_0}{a_3}\right)^{\frac{2}{\varepsilon_1}}\right] = 1 \ . \tag{2.41}$$

From this equation, it follows directly

$$F(x_0, y_0, z_0) = \beta^{-\frac{2}{\varepsilon_1}} . \tag{2.42}$$

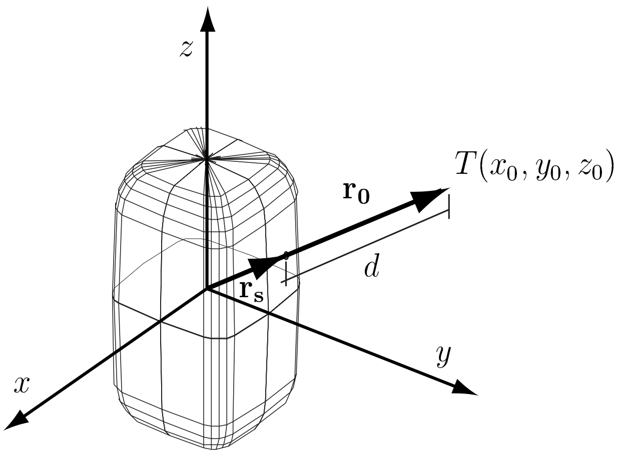


Figure 2.9. Geometric interpretation of the radial Euclidean distance

Thus the radial Euclidean distance is

$$\begin{aligned} d = |\mathbf{r_0} - \mathbf{r_S}| = |\mathbf{r_0} - \beta\mathbf{r_0}| &= |\mathbf{r_0}||1 - F^{-\frac{\varepsilon_1}{2}}(x_0, y_0, z_0)| = \\ &= |\mathbf{r_S}||F^{\frac{\varepsilon_1}{2}}(x_0, y_0, z_0) - 1|. \end{aligned} \quad (2.43)$$

So for any point T in space, with given coordinates (x_0, y_0, z_0), we can determine its position relative to the superellipsoid by simply calculating the value of the $F(x_0, y_0, z_0)$. The following properties hold:

- $F(x_0, y_0, z_0) = 1 \iff \beta = 1 \iff$ point T belongs to the surface of the superellipsoid,
- $F(x_0, y_0, z_0) > 1 \iff \beta < 1 \iff$ point T is outside the superellipsoid,
- $F(x_0, y_0, z_0) < 1 \iff \beta > 1 \iff$ point T is inside the superellipsoid.

2.4.3 RIM OF A SUPERELLIPSOID IN GENERAL ORIENTATION

The rim is a closed space curve which partitions the object surface into a visible and invisible part. We would like to find the analytical form of this curve for a superellipsoid in general orientation. Assuming that we look at the superellipsoid from an infinitely distant point (orthographic projection) in the direction of the z axis of the world coordinate system. For the superellipsoid, the point is on the rim if and only if, the viewing unit vector $\mathbf{v} = (0, 0, 1)$ is perpendicular to the surface normal vector

$$\mathbf{v} \cdot \mathbf{n}(\eta, \omega) = 0, \quad (2.44)$$

with the restriction that the surface normal vector length is not equal to 0. To simplify the calculation we use $\mathbf{n}_d(\eta, \omega)$ instead of $\mathbf{n}(\eta, \omega)$. The problem can be solved in two ways: either we express the viewing vector in the local coordinate system of the superellipsoid, or we transform the surface normal vector given in the local coordinate system to the world coordinate system. In both cases we end up with the following equation

$$\frac{n_z}{a_1} \cos^{2-\varepsilon_1}\eta \cos^{2-\varepsilon_2}\omega + \frac{o_z}{a_2} \cos^{2-\varepsilon_1}\eta \sin^{2-\varepsilon_2}\omega + \frac{a_z}{a_3} \sin^{2-\varepsilon_1}\eta = 0. \quad (2.45)$$

First, we will examine the case when η is not equal to 0. We divide the equation above with $\sin^{2-\varepsilon_1}\eta$ and obtain the solution for η

$$\eta(\omega) = \arctan \left[\left(-\frac{a_3}{a_z} \left(\frac{n_z}{a_1} \cos^{2-\varepsilon_2}\omega + \frac{o_z}{a_2} \sin^{2-\varepsilon_2}\omega \right) \right)^{\frac{1}{2-\varepsilon_1}} \right] \quad . \quad (2.46)$$

Note that the arctan function is restricted to the main branch, namely $-\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}$. If η equals 0 in the solution of equation (2.45), then

$$\frac{n_z}{a_1} \cos^{2-\varepsilon_2}\omega + \frac{o_z}{a_2} \sin^{2-\varepsilon_2}\omega = 0 \quad . \quad (2.47)$$

Observing that the restricted arctan function has the value 0, if and only if the argument is equal to 0, we conclude that the equation (2.46) can be used for any $-\pi \leq \omega < \pi$ where the rim is

$$\mathbf{r}(\omega) = \mathbf{r}(\eta(\omega), \omega) \ . \quad (2.48)$$

Orthographic projection of this rim to the (xy)-plane is the occluding contour of the superellipsoid in general position.

2.4.4 AREA OF A SUPERELLIPSE

A superellipse is a parameterized planar curve, defined as

$$\begin{array}{rcl} x & = & a \cos^{\varepsilon_2}\omega \\ y & = & b \sin^{\varepsilon_2}\omega \end{array} \qquad -\pi \leq \omega < \pi \quad . \quad (2.49)$$

The easiest way of finding the area of superellipse is to use the Green formula

$$A = \tfrac{1}{2} \oint_{C'} (x\, dy - y\, dx), \quad (2.50)$$

where the integration path C' is the curve itself. To simplify the integral, we do not choose the curve C', but rather the curve C that consists of the three segments C_1, C_2 and C_3 (Fig. 2.10).

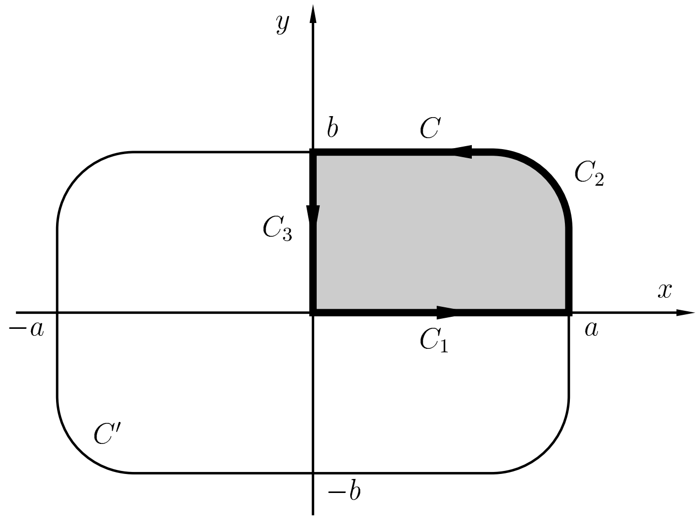


Figure 2.10. Integration path C used in Green formula to calculate the area of superellipse.

Since the superellipse is symmetric with respect to the x and y axes, it follows directly that the area equals

$$\begin{aligned} A &= 2\oint_C (x\,dy - y\,dx) = \\ &= 2\int_{C_1} (x\,dy - y\,dx) + 2\int_{C_2} (x\,dy - y\,dx) + 2\int_{C_3} (x\,dy - y\,dx). \end{aligned} \tag{2.51}$$

The integral along C_1 is equal to 0, because $y = 0$ and $dy = 0$. Similarly, the integral along C_3 is also equal to 0. So now we have

$$\begin{aligned} A &= 2\int_{C_2} (x\,dy - y\,dx) = \\ &= 2\int_0^{\pi/2} (x\dot{y} - y\dot{x})\,d\omega = \\ &= 2ab\varepsilon_2 \int_0^{\pi/2} (\sin^{\varepsilon_2 - 1}\omega \cos^{\varepsilon_2 + 1}\omega \;+ \sin^{\varepsilon_2 + 1}\omega \cos^{\varepsilon_2 - 1}\omega)\,d\omega = \\ &= ab\varepsilon_2 \left[B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2 + 2}{2}\right) + B\left(\frac{\varepsilon_2 + 2}{2}, \frac{\varepsilon_2}{2}\right) \right] = \\ &= 2ab\varepsilon_2\, B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2 + 2}{2}\right) \quad . \end{aligned} \tag{2.52}$$

The term $B(x,y)$ is a beta function and is related to gamma function and defined as

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\phi \cos^{2y-1}\phi \, d\phi = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \qquad (2.53)$$

To verify the result, we will calculate the area for the superellipse with $\varepsilon_2 = 0$ (rectangle), $\varepsilon_2 = 1$ (ellipse) and $\varepsilon_2 = 2$ (deltoid). The last two cases are easy. We get πab for ellipse and $2ab$ for the deltoid. The first case, $\varepsilon_2 = 0$, cannot be calculated directly. We have to find the limit, using $\Gamma(x+1) = x\,\Gamma(x)$ and $\Gamma(x) = \Gamma(x+1)/x$

$$\begin{aligned}\lim_{\varepsilon_2 \to 0} \varepsilon_2 \, B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2+2}{2}\right) &= \lim_{\varepsilon_2 \to 0} \varepsilon_2 \frac{\Gamma(\frac{\varepsilon_2+2}{2})\Gamma(\frac{\varepsilon_2+2}{2})}{\frac{\varepsilon_2}{2}\Gamma(\varepsilon_2+1)} = \\ &= \lim_{\varepsilon_2 \to 0} 2 \frac{\Gamma(\frac{\varepsilon_2+2}{2})\Gamma(\frac{\varepsilon_2+2}{2})}{\Gamma(\varepsilon_2+1)} = 2. \qquad (2.54)\end{aligned}$$

So the area of a rectangular superellipse equals $4ab$.

2.4.5 VOLUME OF A SUPERELLIPSOID

If we cut a superellipsoid with a plane parallel to the (xy)-plane, we get a superellipse (Fig. 2.11).

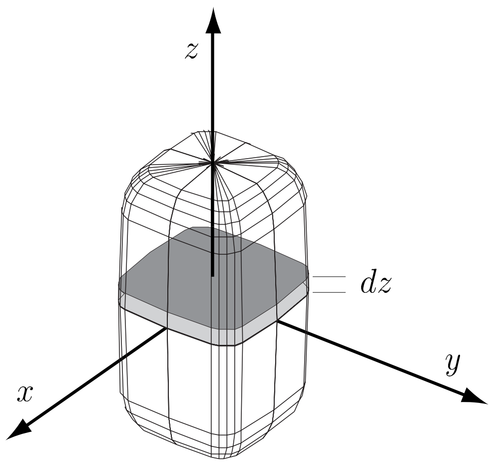


Figure 2.11. Geometric interpretation of a superellipsoid as a stack of superellipses with infinitesimal thickness dz, their size being modulated by another superellipse.

What are the parameters of this superellipse? The parameters a and b depend on the distance of the plane from the origin of the coordinate system, that is the z coordinate. The z coordinate in turn depends only on the parameter η. The area of superellipse thus equals

$$A(\eta) = 2a(\eta)b(\eta)\varepsilon_2\, B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2+2}{2}\right), \tag{2.55}$$

where

$$\begin{aligned} a(\eta) &= a_1 \cos^{\varepsilon_1}\eta, && (2.56)\\ b(\eta) &= a_2 \cos^{\varepsilon_1}\eta. && (2.57) \end{aligned}$$

The corresponding volume differential follows

$$\begin{aligned} dV &= A(z)\, dz = \\ &= A(\eta)\dot{z}(\eta)\, d\eta = \\ &= 2a_1a_2a_3\varepsilon_1\varepsilon_2\, B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2+2}{2}\right) \sin^{\varepsilon_1-1}\eta \cos^{2\varepsilon_1+1}\eta\, d\eta. \quad (2.58) \end{aligned}$$

We will again use the property of superellipsoid symmetry with respect to the (xy)-plane to calculate the volume of a superellipsoid. The integration interval is from 0 to a_3 with respect to z or from 0 to $\pi/2$ with respect to η,

$$\begin{aligned} V &= 2\int_0^{a_3} A(z)\, dz = \\ &= 2\int_0^{\pi/2} A(\eta)\dot{z}(\eta)\, d\eta = \\ &= 4a_1a_2a_3\varepsilon_1\varepsilon_2\, B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2+2}{2}\right) \int_0^{\pi/2} \sin^{\varepsilon_1-1}\eta \cos^{2\varepsilon_1+1}\eta\, d\eta = \\ &= 2a_1a_2a_3\varepsilon_1\varepsilon_2\, B\left(\frac{\varepsilon_1}{2}, \varepsilon_1+1\right) B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2+2}{2}\right) \quad . \quad (2.59) \end{aligned}$$

By algebraic manipulation of the beta terms expressed as gamma functions, we can derive the alternative form

$$V = 2a_1a_2a_3\varepsilon_1\varepsilon_2\, B\left(\frac{\varepsilon_1}{2}+1, \varepsilon_1\right) B\left(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2}{2}\right). \tag{2.60}$$

Fig. 2.12 shows the dependence of the ratio between the volume of a superellipsoid and a parallelepiped, with the sides $2a_1$, $2a_2$ in $2a_3$, on the shape parameters ε_1 in ε_2. Verification of the derived formula for the volume of the superellipsoids shown in Fig. 2.5 produced the values given in Table 2.1.

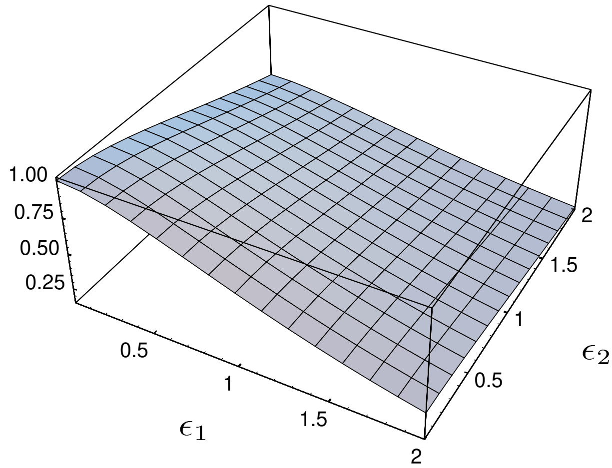


Figure 2.12. Graph of the function $\frac{1}{4}\varepsilon_1\varepsilon_2 B(\frac{\varepsilon_1}{2}, \varepsilon_1+1)B(\frac{\varepsilon_2}{2}, \frac{\varepsilon_2+2}{2})$ which shows how volume of superellipsoids depends on ε_1 and ε_2.

Table 2.1. Volumes for the family of the superquadrics shown in Fig. 2.5 and calculated by equation (2.59).

	$\varepsilon_1 = 0$	$\varepsilon_1 = 1$	$\varepsilon_1 = 2$
$\varepsilon_2 = 0$	$8a_1a_2a_3$	$\frac{16}{3}a_1a_2a_3$	$\frac{8}{3}a_1a_2a_3$
$\varepsilon_2 = 1$	$2\pi a_1a_2a_3$	$\frac{4}{3}\pi a_1a_2a_3$	$\frac{2}{3}\pi a_1a_2a_3$
$\varepsilon_2 = 2$	$4a_1a_2a_3$	$\frac{8}{3}a_1a_2a_3$	$\frac{4}{3}a_1a_2a_3$

2.4.6 MOMENTS OF INERTIA OF A SUPERELLIPSE

As we have used the expression for area of a superellipse to derive the volume of a superellipsoid, we will use expressions for moments of inertia of a superellipse to derive expressions for moments of inertia of a superellipsoid. To simplify the calculation of the moments of inertia, we introduce a "superelliptical" coordinate system similar to a circular coordinate system with coordinates r and ω instead of x and y, where the transformation between the two systems is given by

$$\begin{aligned} x &= ar\cos^{\varepsilon}\omega, \\ y &= br\sin^{\varepsilon}\omega. \end{aligned} \tag{2.61}$$

The determinant of Jacobian matrix for the transformation equals

$$|\mathbf{J}| = abr\varepsilon\sin^{\varepsilon-1}\omega\cos^{\varepsilon-1}\omega. \tag{2.62}$$

and the moments of inertia of a superellipse about the x, y and z axes are as follows

$$\begin{aligned} I^0_{xx} &= \int\int_S y^2 dxdy = \\ &= \int_{-\pi}^{\pi}\int_0^1 b^2r^2 \sin^{2\varepsilon}\omega |\mathbf{J}| dr d\omega = \\ &= \frac{1}{2}ab^3\varepsilon B\left(\frac{3\varepsilon}{2}, \frac{\varepsilon}{2}\right), \end{aligned} \tag{2.63}$$

$$\begin{aligned} I^0_{yy} &= \int\int_S x^2 dxdy = \\ &= \int_{-\pi}^{\pi}\int_0^1 a^2r^2 \cos^{2\varepsilon}\omega |\mathbf{J}| dr d\omega = \\ &= \frac{1}{2}a^3b\varepsilon B\left(\frac{3\varepsilon}{2}, \frac{\varepsilon}{2}\right), \end{aligned} \tag{2.64}$$

$$\begin{aligned} I^0_{zz} &= \int\int_S (x^2+y^2) dxdy = \\ &= I^0_{xx} + I^0_{yy} = \\ &= ab(a^2+b^2)\varepsilon B\left(\frac{3\varepsilon}{2}, \frac{\varepsilon}{2}\right). \end{aligned} \tag{2.65}$$

where $B(x, y)$ is a beta function. The symmetry of a superellipse with respect to the x and y axes of coordinate system causes the moment of deviation to vanish

$$I^0_{xy} = \int\int_S xy\, dxdy = 0. \tag{2.66}$$

The evaluation of the expressions (2.63), (2.64), and (2.65) for a circle, an ellipse, and a rectangle, taking limits where necessary, produces expected results listed in Table 2.2.

Table 2.2. Moments of inertia for special cases of superellipses

	Circle ($\varepsilon = 1$)	*Ellipse* ($\varepsilon = 1$)	*Rectangle* ($\varepsilon = 0$)
I^0_{xx}	$\frac{\pi}{4}r^4$	$\frac{\pi}{4}ab^3$	$\frac{4}{3}ab^3$
I^0_{yy}	$\frac{\pi}{4}r^4$	$\frac{\pi}{4}a^3b$	$\frac{4}{3}a^3b$
I^0_{zz}	$\frac{\pi}{2}r^4$	$\frac{\pi}{4}ab(a^2+b^2)$	$\frac{4}{3}ab(a^2+b^2)$

2.4.7 MOMENTS OF INERTIA OF A SUPERELLIPSOID

By slicing the superellipsoid along the z axis into slices of infinitesimal thickness dz parallel to xy plane and using Steiner's formula, moments of inertia of a superellipsoid can be determined

$$\begin{aligned}
I_{xx} &= \int\int\int_V (y^2+z^2)dxdydz = \qquad (2.67)\\
&= \int_{-a_3}^{+a3} (\int\int_{S(z)} y^2 dxdy + \int\int_{S(z)} z^2 dxdy)dz = \\
&= \int_{-a_3}^{+a3} (I_{xx}^0(z) + z^2 A(z))dz = \\
&= \int_{-\pi/2}^{+\pi/2} (I_{xx}^0(\eta) + z^2(\eta)A(\eta))\dot{z}(\eta)d\eta = \\
&= \frac{1}{2}a_1a_2a_3\varepsilon_1\varepsilon_2(a_2^2 B(\frac{3}{2}\varepsilon_2, \frac{1}{2}\varepsilon_2)B(\frac{1}{2}\varepsilon_1, 2\varepsilon_1+1) + \\
&\quad +4a_3^2 B(\frac{1}{2}\varepsilon_2, \frac{1}{2}\varepsilon_2+1)B(\frac{3}{2}\varepsilon_1, \varepsilon_1+1)),
\end{aligned}$$

$$\begin{aligned}
I_{yy} &= \int\int\int_V (x^2+z^2)dxdydz = \qquad (2.68)\\
&= \int_{-a_3}^{+a3} (\int\int_{S(z)} x^2 dxdy + \int\int_{S(z)} z^2 dxdy)dz = \\
&= \int_{-a_3}^{+a3} (I_{yy}^0(z) + z^2 A(z))dz = \\
&= \int_{-\pi/2}^{+\pi/2} (I_{yy}^0(\eta) + z^2(\eta)A(\eta))\dot{z}(\eta)d\eta = \\
&= \frac{1}{2}a_1a_2a_3\varepsilon_1\varepsilon_2(a_1^2 B(\frac{3}{2}\varepsilon_2, \frac{1}{2}\varepsilon_2)B(\frac{1}{2}\varepsilon_1, 2\varepsilon_1+1) + \\
&\quad +4a_3^2 B(\frac{1}{2}\varepsilon_2, \frac{1}{2}\varepsilon_2+1)B(\frac{3}{2}\varepsilon_1, \varepsilon_1+1)),
\end{aligned}$$

$$\begin{aligned}
I_{zz} &= \int\int\int_V (x^2+y^2)dxdydz = \qquad (2.69)\\
&= \int_{-a_3}^{+a3} (\int\int_{S(z)} (x^2+y^2)dxdy)dz = \\
&= \int_{-a_3}^{+a3} I_{zz}^0(z)dz =
\end{aligned}$$

$$= \int_{-\pi/2}^{+\pi/2} I_{zz}^0(\eta)\dot{z}(\eta)d\eta =$$
$$= \frac{1}{2}a_1a_2a_3\varepsilon_1\varepsilon_2(a_1^2+a_2^2)B(\frac{3}{2}\varepsilon_2, \frac{1}{2}\varepsilon_2)B(\frac{1}{2}\varepsilon_1, 2\varepsilon_1+1).$$

where $I_{xx}^0(\eta)$, $I_{yy}^0(\eta)$, $I_{zz}^0(\eta)$, and $A(\eta)$ are the respective moments of inertia and the area of a superellipse slice with parameters $a(\eta)$ and $b(\eta)$ given by equations (2.56) and (2.57).

The evaluation of the functions for inertial moments of a superellipsoid produced results in accordance with the well-known expressions for inertial moments of common geometric bodies like a sphere, an ellipsoid, and a cube. The results are listed in Table 2.3.

Table 2.3. Moments of inertia for special cases of superellipsoids

	Sphere	*Ellipsoid*	*Cube*
	$\varepsilon_1 = 1$, $\varepsilon_2 = 1$	$\varepsilon_1 = 1$, $\varepsilon_2 = 1$	$\varepsilon_1 = 0$, $\varepsilon_2 = 0$
I_{xx}	$\frac{8\pi}{15}r^5$	$\frac{4\pi}{15}abc(b^2+c^2)$	$\frac{1}{12}abc(b^2+c^2)$
I_{yy}	$\frac{8\pi}{15}r^5$	$\frac{4\pi}{15}abc(a^2+c^2)$	$\frac{1}{12}abc(a^2+c^2)$
I_{zz}	$\frac{8\pi}{15}r^5$	$\frac{4\pi}{15}abc(a^2+b^2)$	$\frac{1}{12}abc(a^2+b^2)$

2.5 COMPUTATION AND RENDERING OF SUPERQUADRICS

In any implementation using superquadrics one must be careful about the numerical evaluation of superquadric equations. All exponential terms in the implicit superquadric equations are of the form x^{2r}, where r can be any positive real number. In numerical computations one must take care of the correct order of evaluation of these exponential terms and compute them as $(x^2)^r$ to assure that the result is not a complex number but a real one when $x < 0$!

In the explicit superquadric equations one should assume that any exponentiation represents in fact, a *signed power function*

$$x^p = \mathrm{sign}(x)|x|^p = \begin{cases} |x|^p & x \geq 0 \\ -|x|^p & x < 0 \end{cases} .$$

This detail is often missing in articles related to superquadrics, but is crucial for any software implementation.

For applications in computer vision, the values for ε_1 and ε_2 are normally bounded: $0 < \{\varepsilon_1, \varepsilon_2\} < 2$, so that only convex shapes are produced (Fig. 2.5). To prevent numerical overflow and difficulties with singularities, ε_1 and ε_2 are often further bounded $(0.1 < \{\varepsilon_1, \varepsilon_2\} < 1.9)$.

Wire-frame models are normally used for rendering superquadrics in computer vision. Rendering accuracy can be controlled simply by changing the sampling rate of the chosen parameterization. Hidden surfaces can be removed by checking the normal vectors which are easy to compute since they are dual to the surface vector (Eq. 2.39). Using the surface normal vector one can also easily generate shaded superquadric models (Fig. 2.13).

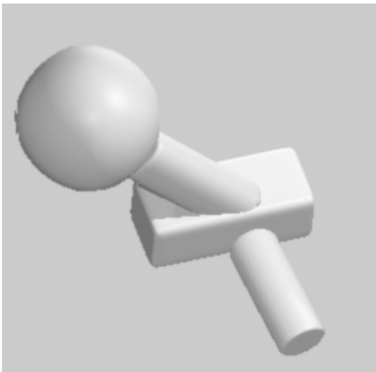


Figure 2.13. A shaded superquadric model using the Phong illumination model and three point sources of illumination.

If the angle parameters η and ω in the explicit equation (2.10) are uniformly sampled, we obtain a "trigonometric" parameterization. With trigonometric parameterization the lines of the wire-frame models are closer together in the more curved regions. This is a good feature for wire-frame models since it gives a good indication of the curvature of the model's surface. For some computer graphics applications such parameterization does not always produce satisfying results. Texturing of superquadric shapes, for example, requires a more uniform parameterization density of the surface. Other types of parameterization of superquadric surfaces were proposed which resulted in an almost uniform parameterization (Franklin and Barr, 1981; Löffelmann and Gröller, 1994; Montiel et al., 1998).

A uniform and dense parameterization of a superquadric surface can be obtained if the explicit equation, where z is as a function of x and y,

$$z = a_3 \left(1 - \left(\left(\frac{x}{a_1} \right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2} \right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{\varepsilon_1}} \right)^{\frac{\varepsilon_1}{2}} \tag{2.70}$$

is expressed as a binomial expansion (Franklin and Barr, 1981). Since only up to the first five coefficients of the expansion need to be computed for a high resolution display, one can easily evaluate equation (2.70) for every pixel (x, y).

The "Angle, Center" parameterization (Löffelmann and Gröller, 1994) uses the angles η and ω of a ray r through the point $\mathbf{r}(\eta, \omega)$ on the superquadric surface. For a superellipse, for example, a parameterization point is defined as

$$\mathbf{r}(\omega) = r(\omega) \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}, \tag{2.71}$$

where

$$r(\omega) = \frac{1}{\sqrt{\left(\left(\frac{\cos\omega}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{\sin\omega}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right)^{\varepsilon_2}}}. \tag{2.72}$$

Superquadrics can be parameterized in the same way.

A regular distribution of parameters along the superquadric surface can be obtained by treating superquadrics as a deformation of ellipsoids (Montiel et al., 1998). This linear-arc length parameterization has also a lower computational cost since the evaluation of rational exponents is avoided. An equal-distance sampling of superellipse models was also proposed by Pilu and Fisher (Pilu and Fisher, 1995).

Appendix A contains the source code for display of superquadric models in the *Mathematica* software package.

2.6 SUMMARY

In computer vision, shape models are chosen according to their degree of uniqueness and compact representation, their local support, expressiveness, and preservation of information. Superquadrics are an extension of quadric surfaces that can model a large variety of generic shapes which are useful for volumetric part representation of natural and man-made objects.

Superquadrics are defined by the explicit or implicit equation. The implicit form (Eq. 2.15) is important for the recovery of superquadrics and testing for intersection, while the explicit form (Eq. 2.10) is more suitable for rendering. We derived geometric properties such as area and moments of a superellipse and radial Euclidean distance from a point to a superellipsoid, superellipsoid volume and superellipsoid moments of inertia. These properties are useful not only for the recovery of superquadrics, but also for other tasks such as range image registration and object recognition.