LAMÉ SURFACES AS A GENERALISATION OF THE TRIAXIAL ELLIPSOID

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ABSTRACT

In modern geodesy the triaxial ellipsoid as a generalisation of the ellipsoid of revolution has a significant position in studying the figure of the Earth. Lamé surfaces represent a generalisation of the triaxial ellipsoid. The following paragraphs are devoted to curvatures of the Lamé surfaces.

Keywords: triaxial ellipsoid, Lamé surfaces, principal curvatures, Gaussian curvature

1. INTRODUCTION

Let us begin with short historical notes. Fedor' Fedorovich' Shubert (Schubert) [1789–1865; Russian general, head of triangu-lation of the St. Petersburg district, participant in many geodetic and astronomical projects; his father Theodor Schubert (1758–1825), born in Braunschweig, astronomer, member of the Academy in St. Petersburg - see *Èntsiklop. slovar' (1903)*] already in 1859 proposed the triaxial ellipsoid as the figure of the Earth (see *Pizzetti (1906)*; on page 235 there are further data on the first applications of the triaxial ellipsoid in geodesy). For more recent data see *Hopfner (1930)* and *Fondelli (1965)*. *Schmehl (1927)* introduced geographic coordinates on the triaxial ellipsoid and studied this ellipsoid applying methods of differential geometry.

Burša and Pěč (1993), *Burša and Kostelecký (1999)*, *Burša (2001)* use the triaxial ellipsoid for studying the figure of the Earth and other celestial bodies; see *Burša and Šíma (1980)* too.

As an example of especially significant triaxiality is Phobos with $a = 13\,500$ m, $b = 10\,700$ m, $c = 9\,600$ m; see *Burša (1989)*.

Lamé (1818) defined and studied the curves, which also comprise the ellipse as a very special case. Among the Lamé curves are, e.g., the known astroid and evolute of an ellipse. For references see *Loria (1910)*, *Teixeira (1909)*, *Brocard and Lemoyne (1967)*. The extension of the Lamé curves to Lamé surfaces - similarly as the transition from the ellipse to the triaxial ellipsoid - on the basis of analogous analytical expressions is natural. Among the Lamé surfaces the triaxial ellipsoid is a very special case.

The following is devoted to the differential geometry of the Lamé surfaces. However, we shall first show the origin of their analytical representation.

2. PRELIMINARIES

We establish the origin in the plane on the system of orthogonal coordinates x, y, or in space on the system of orthogonal coordinates x, y, z.

In the first quadrant the ellipse with semi-axes a, b

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{1}$$

has the parametric representation

$$x = a^2 \frac{\cos\varphi}{h}\,,\ y = b^2 \frac{\sin\varphi}{h}\,,\ \varphi \in \left\langle 0, \frac{\pi}{2} \right\rangle, \tag{2}$$

where

$$h = \left[\left(a\cos\varphi\right)^2 + \left(b\sin\varphi\right)^2 \right]^{1/2} \tag{3}$$

is the distance of the origin, i.e. the centre of the ellipse, from the tangent at point $P[x,y]$ of the ellipse. In the theory of convex figures this oriented distance is called the support function. The essential part of this theory is based only on studying the support function and its properties. The curvature of ellipse (1) at its point P is

$$k = \frac{1}{a^2 b^2} h^3 \,. \tag{4}$$

For the vertices of this ellipse $h = a$, or $h = b$, and from Eq.(4) follows, on the one hand, b^2/a as the radius of curvature at the vertex on axis x, on the other hand, a^2/b as the radius of curvature at the vertex on axis y. Let us also point out, that, if $a > b$, support function h from (3) is the a-multiple of geodetic function W; see *Kostelecký and Nádeník (1971)*.

In the first quadrant the natural generalisation of ellipse (1) is represented by the Lamé curve

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^p = 1\,,\ p \geq 2\,,\ x \geq 0\,,\ y \geq 0\,. \tag{5}$$

Its parametric representation is

$$x = a^{\frac{p}{p-1}} \left(\frac{\cos\varphi}{h}\right)^{\frac{1}{p-1}},\ y = b^{\frac{p}{p-1}} \left(\frac{\sin\varphi}{h}\right)^{\frac{1}{p-1}}, \tag{6}$$

where

$$h=\left[\left(a\cos\varphi\right)^{\frac{p}{p-1}}+\left(b\sin\varphi\right)^{\frac{p}{p-1}}\right]^{(p-1)/p}, \quad \varphi\in\left\langle 0,\frac{\pi}{2}\right\rangle \tag{7}$$

is the support function of our Lamé curve, relative to the origin. The formulae for the curvature of the curve are well-known from the theory of plane curves. Using these formulae we can see that the curvature of Lamé curve (5)

$$k=\frac{p-1}{a^p b^p}\left(xy\right)^{p-2}h^3 \tag{8}$$

with x, y from (6) and with h from (7). Especially if $p = 2$, (1)–(4) follow from (5)–(8).

In the first octant the ellipsoid with semi-axes a, b, c

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1 \tag{9}$$

has the parametric representation

$$x=a^2\frac{\cos\varphi\cos\lambda}{H}, \quad y=b^2\frac{\cos\varphi\sin\lambda}{H}, \quad z=c^2\frac{\sin\varphi}{H}, \quad 0\le\varphi\le\frac{\pi}{2}, \quad 0\le\lambda\le\frac{\pi}{2}, \tag{10}$$

where

$$H=\left[\left(a\cos\varphi\cos\lambda\right)^2+\left(b\cos\varphi\sin\lambda\right)^2+\left(c\sin\varphi\right)^2\right]^{1/2} \tag{11}$$

is the distance of the origin, i.e. the centre of the ellipsoid, from the tangent plane at point $P[x,y,z]$ of the ellipsoid. In the terminology of the theory of convex bodies H is the support function, the application of which forms the essential part of this theory. The Gaussian curvature of an ellipsoid

$$K=\frac{1}{a^2b^2c^2}H^4, \tag{12}$$

see also *Kostelecký and Nádeník (1971)* and *Holota and Nádeník (1971)*, inclusive of references. Eqs.(9)–(12) give a lucid generalisation of Eqs.(1)–(4).

On the ellipsoid Eq.(12) enables a very simple construction of the curves, along which the Gaussian curvature is constant. We choose a sphere, concentric with the ellipsoid, whose radius H is such that there exist common tangent planes to our ellipsoid and sphere. On the ellipsoid the tangent points of these planes form a curve with constant Gaussian curvature.

Please, note that *Burša and Pěč (1993)*, *Burša and Kostelecký (1999)*, *Burša (2001)* work with the radius vector of point P of the triaxial ellipsoid, but this paper is based on the support function.

The above mentioned formulae, or the way in which they are deduced, are known.

In the first octant the direct generalisation of ellipsoid (9) is the Lamé surface

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^p + \left(\frac{z}{c}\right)^p = 1 \,, \; p \geq 2 \,, \; x \geq 0 \,, \; y \geq 0 \,, \; z \geq 0 \,. \tag{13}$$

The significance of the restricting inequalities for coordinates x, y, z is evident: It is necessary to define their p-powers. The significance of the inequality for exponent p will become clear further on. Only in this remark we admit $p > 0$: if $p = 1$, we obtain a plane (more exactly: its triangle in the first quadrant); if $p \to 0$, it is evident that surface (13) transforms into 3 faces of a rectangular parallelepiped (in the first octant) the edges of which are in the coordinate axes; if $p \to \infty$, it is similarly seen that surface (13) transforms into the 3 remaining faces of that parallelepiped.

The parametric representation of surface (13) reads

$$x = a^{\frac{p}{p-1}} \left(\frac{\cos\varphi \cos\lambda}{H}\right)^{\frac{1}{p-1}} \,,$$

$$y = b^{\frac{p}{p-1}} \left(\frac{\cos\varphi \sin\lambda}{H}\right)^{\frac{1}{p-1}} \,,$$

$$z = c^{\frac{p}{p-1}} \left(\frac{\sin\varphi}{H}\right)^{\frac{1}{p-1}} \,,$$

$$0 \leq \varphi \leq \frac{\pi}{2} \,, \; 0 \leq \lambda \leq \frac{\pi}{2} \,, \tag{14}$$

where

$$H = \left[\left(a \cos\varphi \cos\lambda\right)^{\frac{p}{p-1}} + \left(b \cos\varphi \sin\lambda\right)^{\frac{p}{p-1}} + \left(c \sin\varphi\right)^{\frac{p}{p-1}} \right]^{(p-1)/p} \tag{15}$$

is the support function of our Lamé surface, relative to the origin.

If we also extend the parametric representation to further octants, (14) would have to be modified as shown for $0 \leq \varphi \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq \lambda \leq \pi$:

$$x = -a^{\frac{p}{p-1}} \left(\frac{\cos\varphi \left|\cos\lambda\right|}{H}\right)^{\frac{1}{p-1}}$$

and y and z remain without change.

Eqs.(14) immediately indicate that parametric curve $\lambda =$ const. lies in the plane

$$\left(b^{\frac{p}{p-1}}\sin^{\frac{1}{p-1}}\lambda\right)x-\left(a^{\frac{p}{p-1}}\cos^{\frac{1}{p-1}}\lambda\right)y=0\;.$$

Hence parametric curves λ = const. form an analogue to the meridians of the ellipsoid of revolution. Point $[0,0,c]$ of the Lamé surface is an unsubstantial singularity in parameters φ, λ. Parametric curve φ = const. $\neq 0, \pi/2$ is projected from the origin through the Lamé cone

$$\frac{x^{2p-2}}{a^{2p}\cos^2\varphi}+\frac{y^{2p-2}}{b^{2p}\cos^2\varphi}-\frac{z^{2p-2}}{c^{2p}\sin^2\varphi}=0\;.$$

The analogue with the parallels of the ellipsoid of revolution is by far not as close.

In Section 3 for the Gaussian curvature of Lamé surface (13) we obtain the formula

$$K=\frac{(p-1)^2}{a^p b^p c^p}(xyz)^{p-2}H^4\;, \tag{16}$$

with x, y, z from (14) and with H from (15). Eq.(16) also shows the reason for our restriction $p \geq 2$; if $0 < p < 2$, if $x \to 0$ or $y \to 0$ or $z \to 0$, then $K \to \infty$, i.e. a singularity would occur.

Equations and formulae (13)–(16) are generalisations (from the Lamé curve to the Lamé surface) of equations and formulae (5)–(8) and generalisation (from ellipsoid to the Lamé surface) of equations and formulae (9)–(12) too.

In Section 3 we find an equation for the principal curvatures of the Lamé surface and formulae for its Gaussian curvature K. In Section 4 we calculate components b_{11}, $b_{12} = b_{21}$, b_{22} of the second fundamental tensor and its discriminant

$$B^2=\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{vmatrix}\;. \tag{17}$$

In Section 5 we examine component $a_{12} = a_{21}$ of the first fundamental tensor and its discriminant (a_{11}, a_{22} are unmixed components)

$$A^2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}\;. \tag{18}$$

From differential geometry we know that

$$K=\frac{B^2}{A^2}\;. \tag{19}$$

Burša and Pěč (1993) and *Burša and Kostelecký (1999)* work with these tensors.

3. GAUSSIAN CURVATURE OF THE LAMÉ SURFACE

Let the partial derivatives of the second order of function $F(x,y,z)$ in the first octant be continuous and

$$\Phi^2 = F_x^2 + F_y^2 + F_z^2 > 0 \,. \tag{20}$$

On the surface with equation

$$F(x,y,z) = 0 \tag{21}$$

let us denote the principal curvatures as $1/R_1$ and $1/R_2$. These curvatures are the roots of equation

$$\begin{vmatrix} F_{xx} - \frac{\Phi}{R} & F_{xy} & F_{xz} & F_x \\ F_{yx} & F_{yy} - \frac{\Phi}{R} & F_{yz} & F_y \\ F_{zx} & F_{zy} & F_{zz} - \frac{\Phi}{R} & F_z \\ F_x & F_y & F_z & 0 \end{vmatrix} = 0 \,, \tag{22}$$

(see *Kommerell (1896)*, *Kommerell and Kommerell (1903)*, *Staude (1910)*; this equation has been omitted in contemporary books on metric differential geometry).

For Lamé surface (13)

$$F(x,y,z) = \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^p + \left(\frac{z}{c}\right)^p - 1 = 0 \tag{23}$$

is

$$F_x = \frac{p}{a^p} x^{p-1} \,,\; F_y = \frac{p}{b^p} y^{p-1} \,,\; F_z = \frac{p}{c^p} z^{p-1} \,, \tag{24}$$

$$F_{xx} = \frac{p(p-1)}{a^p} x^{p-2} \,,\; F_{yy} = \frac{p(p-1)}{b^p} y^{p-2} \,,\; F_{zz} = \frac{p(p-1)}{c^p} z^{p-2} \tag{25}$$

$$F_{xy} = 0 \,,\; F_{yz} = 0 \,,\; F_{zx} = 0 \,. \tag{26}$$

We can see that $\Phi = 0$ if and only if $x = y = z = 0$. But due to (23) the origin is not a point of the Lamé surface, hence inequality (20) holds on it.

In the case of (26), Eq.(22) is reduced as follows:

$$\left(\frac{\Phi}{R}\right)^2\left(F_x^2+F_y^2+F_z^2\right)-\frac{\Phi}{R}\left[F_x^2\left(F_{yy}+F_{zz}\right)+F_y^2\left(F_{zz}+F_{xx}\right)+F_z^2\left(F_{xx}+F_{yy}\right)\right]$$
$$+\left(F_x^2F_{yy}F_{zz}+F_y^2F_{zz}F_{xx}+F_z^2F_{xx}F_{yy}\right)=0\,.$$

Hence - with respect to (20) -

$$\frac{1}{R_1}\frac{1}{R_2}=\frac{F_x^2F_{yy}F_{zz}+F_y^2F_{zz}F_{xx}+F_z^2F_{xx}F_{yy}}{\left(F_x^2+F_y^2+F_z^2\right)^2}\,. \tag{27}$$

For the Lamé surface, i.e. in cases (24) and (25), it holds firstly, also due to (14)

$$F_x^2+F_y^2+F_z^2=p^2\left(\frac{x^{2p-2}}{a^{2p}}+\frac{y^{2p-2}}{b^{2p}}+\frac{z^{2p-2}}{c^{2p}}\right)=\frac{p^2}{H^2}\,,$$

and secondly - also due to (23) or (13)

$$F_x^2F_{yy}F_{zz}+F_y^2F_{zz}F_{xx}+F_z^2F_{xx}F_{yy}=\frac{p^4\left(p-1\right)^2}{a^pb^pc^p}x^{p-2}y^{p-2}z^{p-2}\left(\frac{x^p}{a^p}+\frac{y^p}{b^p}+\frac{z^p}{c^p}\right)$$
$$=\frac{p^4\left(p-1\right)^2}{a^pb^pc^p}\left(xyz\right)^{p-2}\,.$$

For Gaussian curvature K of the Lamé surface we thus obtain from (27)

$$K=\frac{\left(p-1\right)^2}{a^pb^pc^p}\left(xyz\right)^{p-2}H^4\,. \tag{28}$$

This is Eq.(16) mentioned earlier.

From it follows immediately: Among our Lamé surfaces the ellipsoid is characterised by this property: the Gaussian curvature depends only on the support function, i. e. not on the product of coordinates xyz. The points of the Lamé surface in the coordinate planes (i.e. points having $x=0$ or $y=0$ or $z=0$) are parabolic if $p>2$ (if $p=2$ elliptic on the ellipsoid).

Gaussian curvature K from (28) can be simply geometrically interpreted. Let us construct the normal at point $P[x,y,z]$ of the Lamé surface (13) and define its intersection N_{xy} with co-ordinate plane $z=0$. We can easily determine that the distance of point P from intersection N_{xy} is

$$\overline{PN_{xy}}=c^p\frac{z^{2-p}}{H}\,.$$

By cyclic permutation we obtain

$$\overline{PN_{yz}}=a^p\frac{x^{2-p}}{H}\,,\ \overline{PN_{zx}}=b^p\frac{y^{2-p}}{H}\,.$$

Hence with respect to (28)

$$k=\left(p-1\right)\frac{H}{\overline{PN_{xy}}\cdot\overline{PN_{yz}}\cdot\overline{PN_{zx}}}\,.$$

Similarly as we calculated and geometrically interpreted Gaussian curvature $K = (1/R_1)(1/R_2)$, we can also calculate and geometrically interpret mean curvature $(1/R_1) + (1/R_2)$.

4. SECOND FUNDAMENTAL TENSOR

In Sections 4 and 5 we restrict ourselves to $0 < \varphi < \pi/2$, $0 < \lambda < \pi/2$.
Unit vector $\underline{\boldsymbol{N}}$ of the normal of Lamé surface (13) or (14) is

$$\underline{\boldsymbol{N}}\left(\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi\right). \tag{29}$$

The co-ordinates of radius vector $\underline{\boldsymbol{x}}(x, y, z)$ of the point of the considered surface are given in (14).

The second fundamental tensor $b_{\alpha\beta}$ (α, $\beta = 1, 2$) is (see *Burša and Pěč (1993)*, *Burša and Kostelecký (1994)* - geodetical literature is referred to, and not the extensive mathematical literature) purposefully

$$b_{11} = \frac{\partial \underline{\boldsymbol{x}}}{\partial\varphi}\frac{\partial \underline{\boldsymbol{N}}}{\partial\varphi}, \; b_{12} = b_{21} = \frac{\partial \underline{\boldsymbol{x}}}{\partial\varphi}\frac{\partial \underline{\boldsymbol{N}}}{\partial\lambda} = \frac{\partial \underline{\boldsymbol{x}}}{\partial\lambda}\frac{\partial \underline{\boldsymbol{N}}}{\partial\varphi}, \; b_{22} = \frac{\partial \underline{\boldsymbol{x}}}{\partial\lambda}\frac{\partial \underline{\boldsymbol{N}}}{\partial\lambda}. \tag{30}$$

We substitute into (30) for $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{N}}$ from (14) and (29). After some algebra (which requires full attention with respect to handling the exponents) we obtain

$$\begin{aligned}
b_{11} &= \frac{c^{\frac{p}{p-1}}}{p-1} H^{\frac{p+1}{-p+1}} \cos^{\frac{-p+2}{p-1}}\varphi \sin^{\frac{-p+2}{p-1}}\varphi \left[a^{\frac{p}{p-1}} \cos^{\frac{p}{p-1}}\lambda + b^{\frac{p}{p-1}} \sin^{\frac{p}{p-1}}\lambda \right], \\
b_{12} = b_{21} &= \frac{c^{\frac{p}{p-1}}}{p-1} H^{\frac{p+1}{-p+1}} \cos^{\frac{1}{p-1}}\varphi \sin^{\frac{1}{p-1}}\varphi \\
&\quad \left[a^{\frac{p}{p-1}} \cos^{\frac{1}{p-1}}\lambda \sin\lambda - b^{\frac{p}{p-1}} \cos\lambda \sin^{\frac{1}{p-1}}\lambda \right], \\
b_{22} &= \frac{1}{p-1} H^{\frac{p+1}{-p+1}} \cos^{\frac{p}{p-1}}\varphi \left\{ a^{\frac{p}{p-1}} \cos^{\frac{-p+2}{p-1}}\lambda \sin^2\lambda \right. \\
&\quad \left[b^{\frac{p}{p-1}} \cos^{\frac{p}{p-1}}\varphi \sin^{\frac{-p+2}{p-1}}\lambda + c^{\frac{p}{p-1}} \sin^{\frac{p}{p-1}}\varphi \right] \\
&\quad \left. + b^{\frac{p}{p-1}} \cos^2\lambda \sin^{\frac{-p+2}{p-1}}\lambda \left[a^{\frac{p}{p-1}} \cos^{\frac{p}{p-1}}\varphi \cos^{\frac{-p+2}{p-1}}\lambda + c^{\frac{p}{p-1}} \sin^{\frac{p}{p-1}}\varphi \right] \right\}.
\end{aligned} \tag{31}$$

With the same diligence we calculate discriminant B^2 from (17), using (15) in conclusion:

$$b^2 = \frac{(abc)^{\frac{p}{p-1}}}{(p-1)^2} H^{\frac{p+2}{-p+1}} \cos^{\frac{2}{p-1}} \varphi \left(\sin\varphi \cos\lambda \sin\lambda\right)^{\frac{-p+2}{p-1}} . \tag{32}$$

5. FIRST FUNDAMENTAL TENSOR

The first fundamental tensor is derived from radius vector $\underline{\boldsymbol{x}}$ with coordinates (14):

$$a_{11} = \frac{\partial \underline{\boldsymbol{x}}}{\partial\varphi}\frac{\partial \underline{\boldsymbol{x}}}{\partial\varphi}, \; a_{12} = a_{21} = \frac{\partial \underline{\boldsymbol{x}}}{\partial\varphi}\frac{\partial \underline{\boldsymbol{x}}}{\partial\lambda}, \; a_{22} = \frac{\partial \underline{\boldsymbol{x}}}{\partial\lambda}\frac{\partial \underline{\boldsymbol{x}}}{\partial\lambda} .$$

The calculation of these scalar products is a technically complicated operation. It is not reproduced here, but only the result for $a_{12} = a_{21}$ is given:

$$\begin{aligned}
a_{12} = a_{21} = {} & \frac{c^{\frac{p}{p-1}}}{(p-1)^2} H^{\frac{2p+2}{-p+1}} \left(\cos^{\frac{2}{p-1}} \varphi \sin^{\frac{1}{p-1}} \varphi \cos\lambda \sin\lambda \right) \\
& \cdot \left[a^{\frac{p}{p-1}} \cos^{\frac{-p+2}{p-1}} \lambda - b^{\frac{p}{p-1}} \sin^{\frac{-p+2}{p-1}} \lambda \right] \\
& \cdot \left\{ (ab)^{\frac{p}{p-1}} \cos^{\frac{1}{p-1}} \varphi \cos^{\frac{-p+2}{p-1}} \lambda \sin^{\frac{-p+2}{p-1}} \lambda \right. \\
& + (bc)^{\frac{p}{p-1}} \left(\cos^{-1} \varphi \sin^{\frac{p}{p-1}} \varphi \sin^{\frac{-p+2}{p-1}} \lambda + \cos\varphi \sin^{\frac{-p+2}{p-1}} \varphi \sin^{\frac{p}{p-1}} \lambda \right) \\
& \left. + (ca)^{\frac{p}{p-1}} \left(\cos^{-1} \varphi \sin^{\frac{p}{p-1}} \varphi \cos^{\frac{-p+2}{p-1}} \lambda + \cos\varphi \sin^{\frac{-p+2}{p-1}} \varphi \cos^{\frac{p}{p-1}} \lambda \right) \right\} .
\end{aligned}$$

If $0 < \varphi < \pi/2$, $0 < \lambda < \pi/2$, then in the formula for $a_{12} = a_{21}$ the first line is positive and the same holds for the factor in brace brackets $\{\ldots\}$ on the third, fourth and fifth lines. Hence $a_{12} = a_{21} = 0$ if, and only if the expression in square brackets $[\ldots]$ on the second line is annulled:

$$a^{\frac{p}{p-1}} \cos^{\frac{-p+2}{p-1}} \lambda - b^{\frac{p}{p-1}} \sin^{\frac{-p+2}{p-1}} \lambda = 0 . \tag{33}$$

This equation can be modified to read:

$$\tan\lambda = \left(\frac{b}{a}\right)^{\frac{p}{p-2}} . \tag{34}$$

If (34) holds - and only in this case, then $a_{12} = a_{21} = 0$. But the geometrical significance of the annulment of the mixed component of the first fundamental tensor is well-known from differential geometry: This means that the parametric curves are orthogonal.

Let us return to the mixed component $b_{12} = b_{21}$ of the second fundamental tensor in (31). We can see that it is annulled if, and only if

$$a^{\frac{p}{p-1}} \cos^{\frac{1}{p-1}} \lambda \sin \lambda - b^{\frac{p}{p-1}} \cos \lambda \sin^{\frac{1}{p-1}} \lambda = 0 \, .$$

Evidently this equation is equivalent to (33). The geometrical significance of the simultaneous annulment of the mixed components of the first and of the second tensors is known from differential geometry, too.This means that the parametric curves are lines of curvature (they have tangents in the direction of extreme normal curvatures).

We can thus say that the parametric curve with λ determined after (34) (we know that each parametric curve $\lambda =$ const. is a plane curve) is a line of curvature. It thus forms an analogue to any meridian of an ellipsoid of revolution.

If we substitute into formula (16) $\equiv$ (28) from (14), i.e. if we pass from coordinates x, y, z to parameters φ, λ, we obtain

$$K = \frac{(p-1)^2}{(abc)^{\frac{p}{p-1}}} H^{\frac{p+2}{p-1}} \cos^{\frac{p-2}{p-1}} \varphi \left(\cos\varphi \sin\varphi \cos\lambda \sin\lambda\right)^{\frac{p-2}{p-1}} .$$

We remind the reader of (19) and by means of (32) we arrive at

$$A^2 = \frac{B^2}{K} = \frac{B^4}{\cos^2 \varphi} \, . \tag{35}$$

The known formulae of the differential geometry of surfaces (see *Hlavatý, 1939; Kreyszig, 1991; Stoker, 1989*) also yield relation $A^2 = B^2/K$ directly.

We begin with the Weingarten equations

$$\frac{\partial \underline{\underline{x}}}{\partial \varphi} = \frac{a_{12}b_{12} - a_{11}b_{22}}{B^2} \frac{\partial \underline{\underline{N}}}{\partial \varphi} + \frac{a_{11}b_{12} - a_{12}b_{11}}{B^2} \frac{\partial \underline{\underline{N}}}{\partial \lambda} ,$$

$$\frac{\partial \underline{\underline{x}}}{\partial \lambda} = \frac{a_{22}b_{12} - a_{12}b_{22}}{B^2} \frac{\partial \underline{\underline{N}}}{\partial \varphi} + \frac{a_{12}b_{12} - a_{22}b_{11}}{B^2} \frac{\partial \underline{\underline{N}}}{\partial \lambda}$$

(using the symbolics of the tensor calculus enables to write these relations more briefly and better arranged). In view of (29)

$$\frac{\partial \underline{\underline{N}}}{\partial \varphi} \cdot \frac{\partial \underline{\underline{N}}}{\partial \varphi} = 1 \, , \; \frac{\partial \underline{\underline{N}}}{\partial \varphi} \cdot \frac{\partial \underline{\underline{N}}}{\partial \lambda} = 0 \, , \; \frac{\partial \underline{\underline{N}}}{\partial \lambda} \cdot \frac{\partial \underline{\underline{N}}}{\partial \lambda} = \cos^2 \varphi \, .$$

The scalar product of the above-mentioned equations with $\partial \underline{\boldsymbol{N}}/\partial\varphi$ and $\partial \underline{\boldsymbol{N}}/\partial\lambda$, with respect to (30), yield

$$b_{11} = \frac{a_{12}b_{12} - a_{11}b_{22}}{B^2}\ , \ b_{12} = \frac{a_{11}b_{12} - a_{12}b_{11}}{B^2}\cos^2\varphi\ ,$$

$$b_{21} = \frac{a_{22}b_{12} - a_{12}b_{22}}{B^2}\ , \ b_{22} = \frac{a_{12}b_{12} - a_{22}b_{11}}{B^2}\cos^2\varphi\ .$$

We can easily check by multiplication that

$$B^2 = b_{11}b_{22} - b_{12}b_{21} = \frac{\cos^2\varphi}{B^4}\left(a_{11}a_{22} - a_{12}a_{21}\right)\left(b_{11}b_{22} - b_{12}b_{21}\right) = \frac{A^2}{B^2}\cos^2\varphi\ ,$$

hence (35) again.

6. CONCLUSIONS

Baeschlin (1948, p.27) wrote (in German): The Earth's ellipsoid proved as a flattend rotational ellipsoid in so great approximation that today is no matter for practisising any geodetic calculation on the triaxial ellipsoid. The mean square error of the today numerically found ellipticity of the Earth's equator is greater than this value itself.

After approximately 50 years, *Burša et al. (1993,1999,2001)* work with the triaxial ellipsoid.

Should the possibility appear, after several decades again to substitute the triaxial ellipsoid by a more suitable reference surface, the Lamé surfaces close to the triaxial ellipsoid (i.e. for $p > 2$ close to 2), would be eligible.

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