Splines on the Sphere
(A View from the Other Hemisphere)

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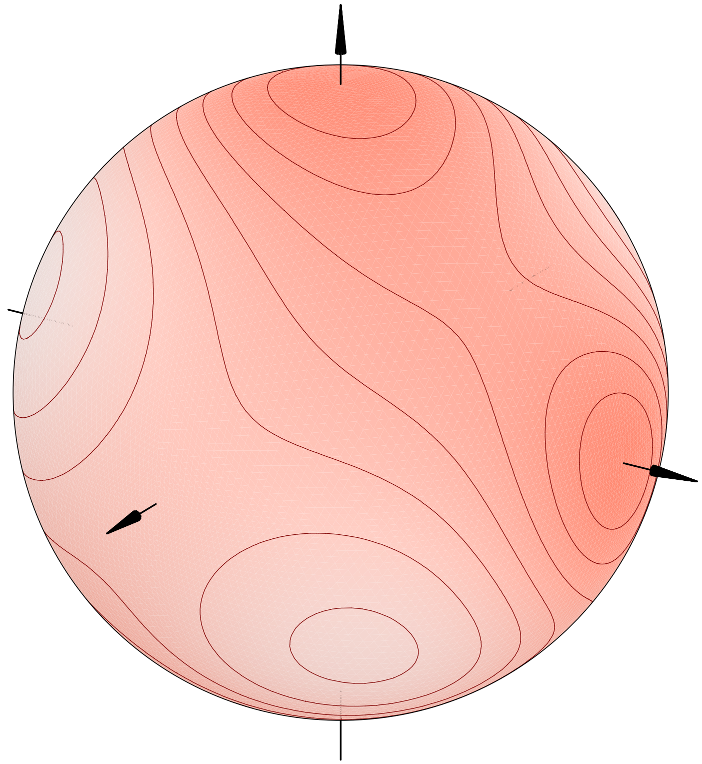
Joint work with

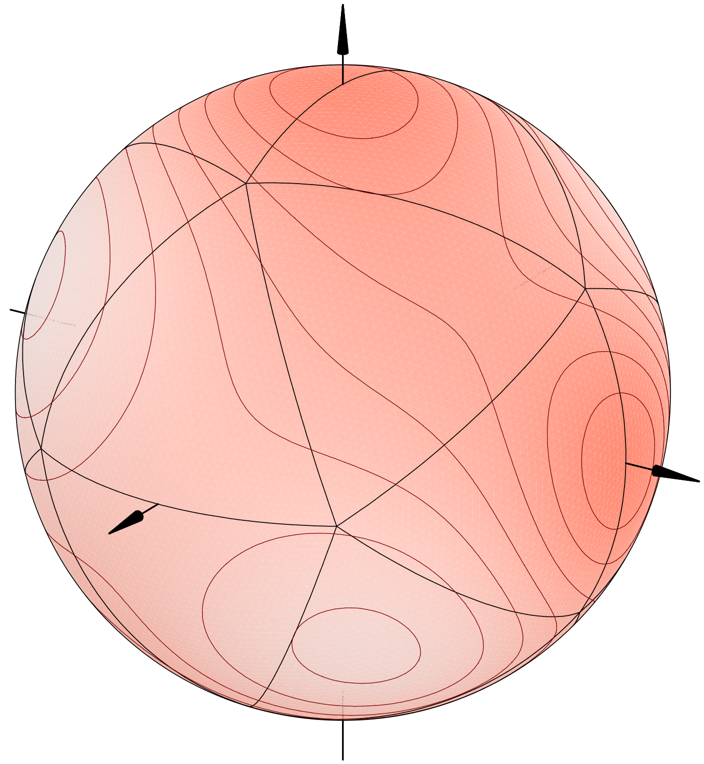
Anamaria Gomide

Spherical Functions and Spherical Splines

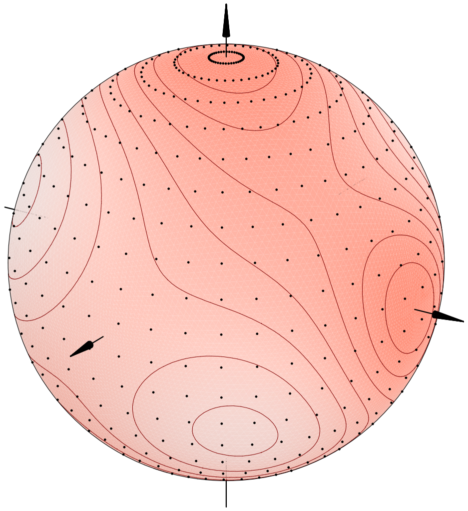
A Spherical function:

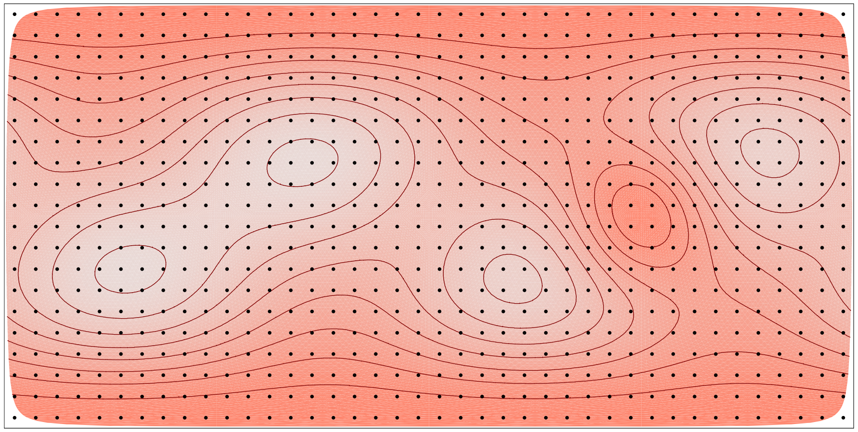


A Spherical spline:

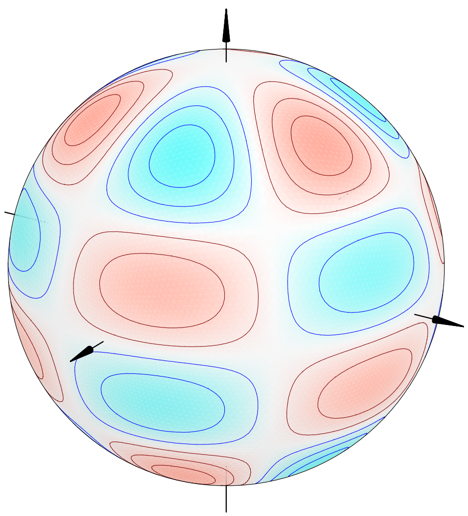


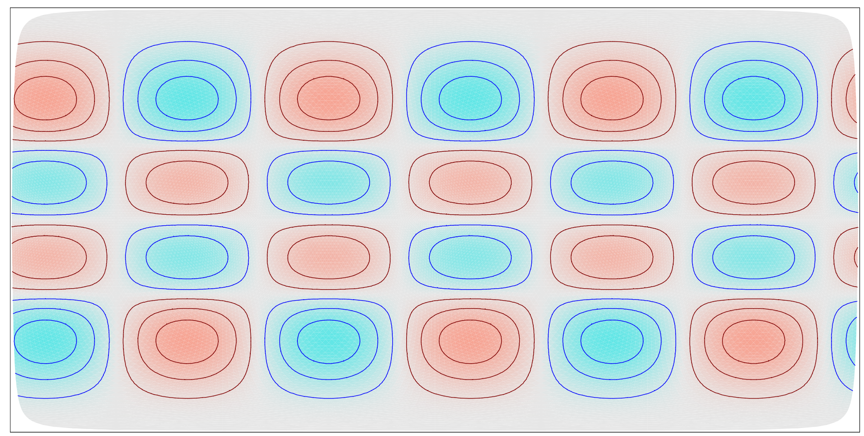
Modeling with Uniform Lon-Lat Grids



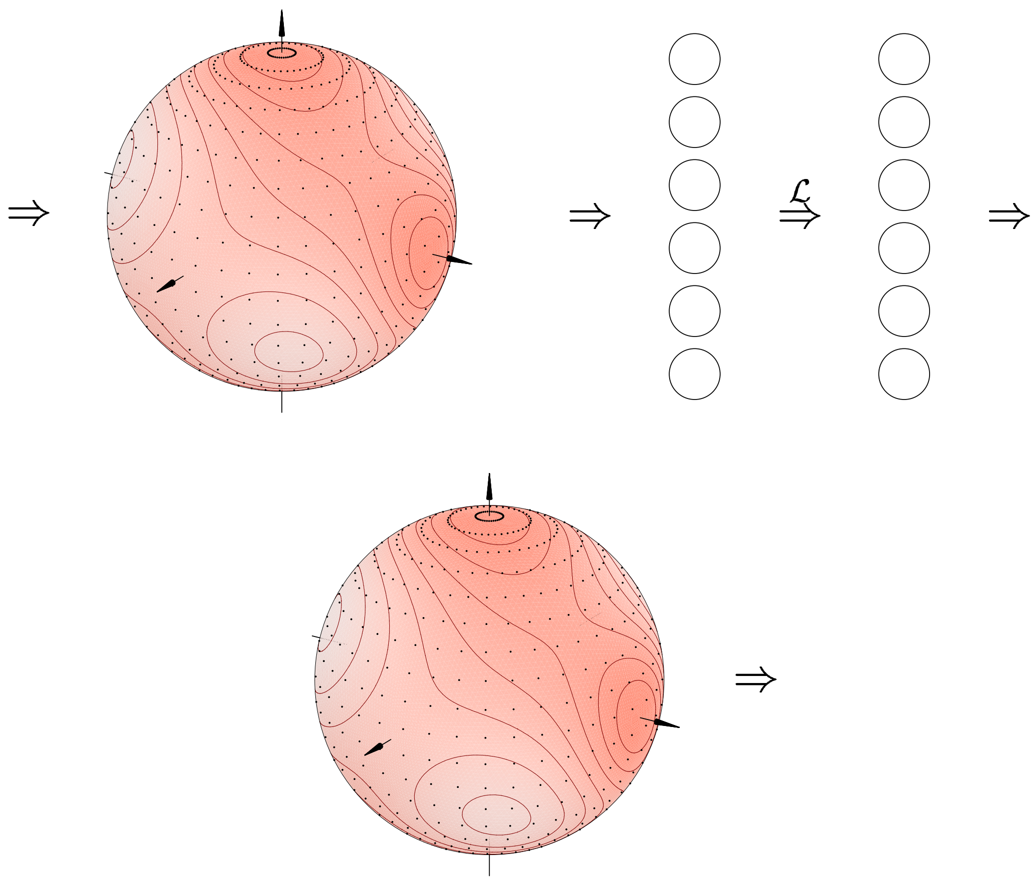


Modeling with Spherical Harmonics



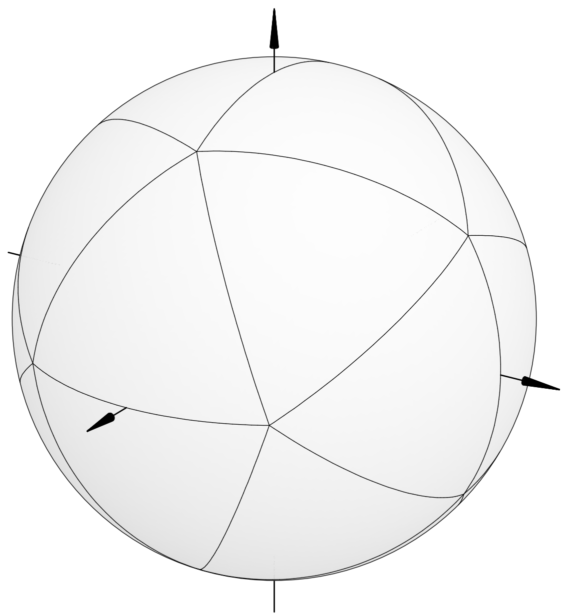


The Spectral Method

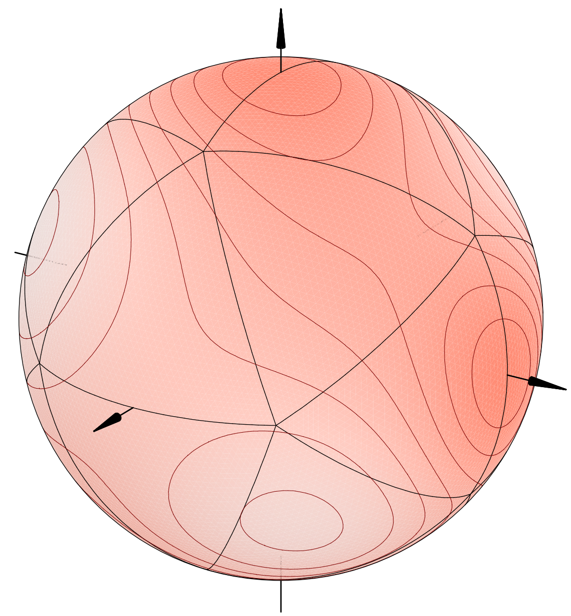


Spherical Splines

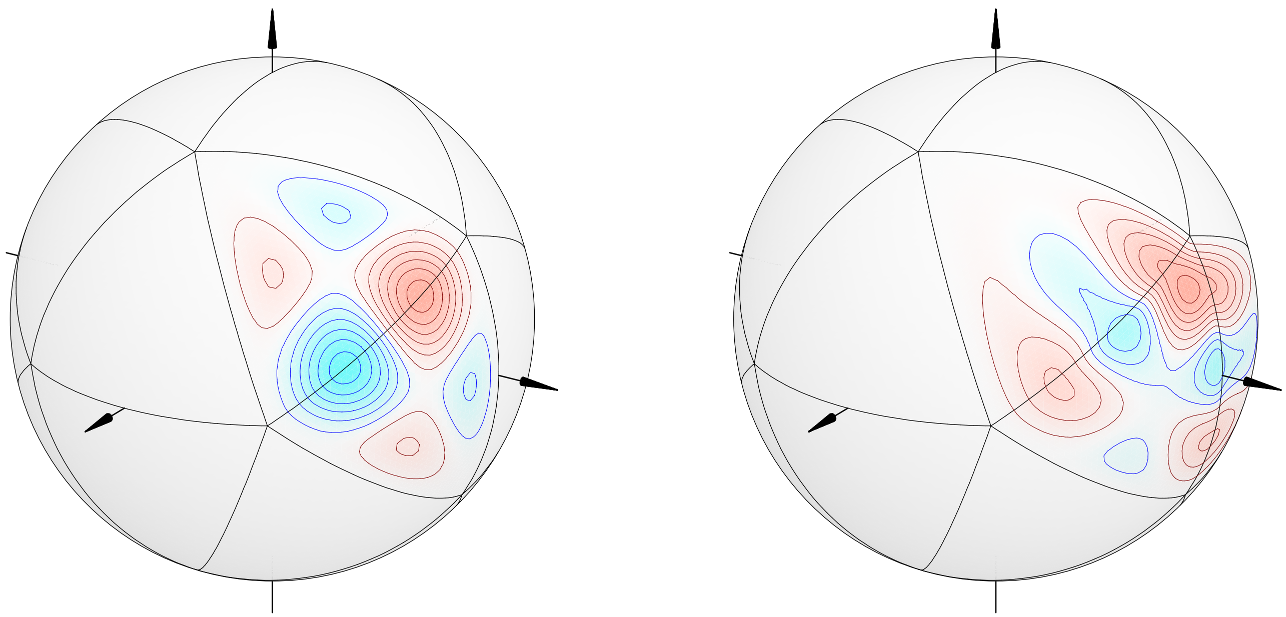
Geodesic triangulation



Spherical spline



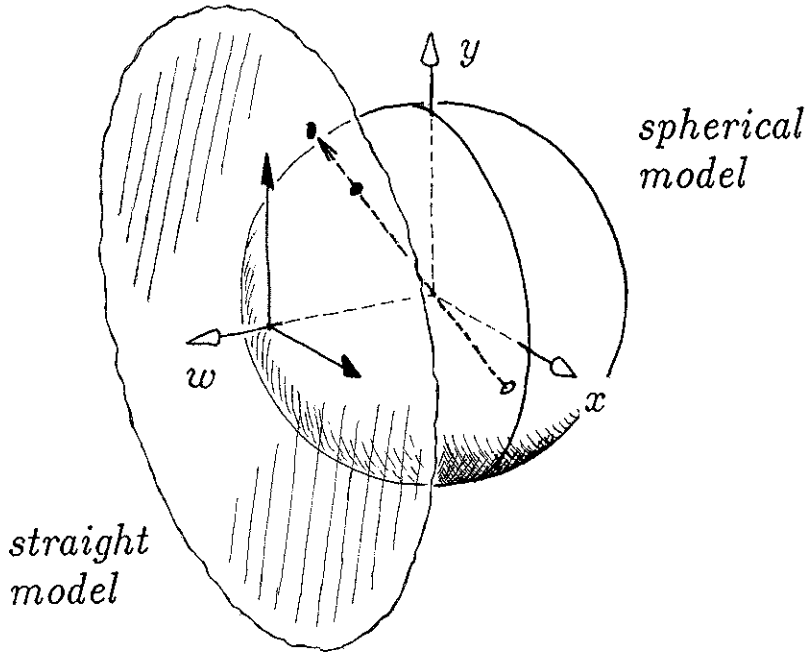
Finite Element Bases for Spherical Splines



Cartesian Approach to Spherical Functions

$F(x, y, z)$

$f = F|\mathbf{S}^2$



Spherical Polynomials

$$\mathcal{H}^d = \text{Homogeneous trivariate polynomials of degree } d.$$

$$\begin{aligned}\mathcal{P}^d &= \text{General trivariate polynomials of degree at most } d \\ &= \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^d\end{aligned}$$

$$\text{On the sphere,} \quad 1 = x^2 + y^2 + z^2.$$

$$\text{Therefore} \quad \mathcal{P}^d|\mathbf{S}^2 = \mathcal{H}^d|\mathbf{S}^2 \oplus \mathcal{H}^{d-1}|\mathbf{S}^2.$$

Spherical Splines

$$\mathcal{H}_r^d[T]|\mathbf{S}^2 = \text{Homogeneous } \mathsf{C}_r \text{ spherical splines on } T \text{ with degree } d.$$

$$\begin{aligned}\mathcal{P}_r^d[T]|\mathbf{S}^2 &= \text{General } \mathsf{C}_r \text{ spherical splines on } T \text{ with degree at most } d \\ &= \mathcal{H}_r^d[T]|\mathbf{S}^2 \oplus \mathcal{H}_r^{d-1}[T]|\mathbf{S}^2\end{aligned}$$

[Gomide and Stolfi 1998]

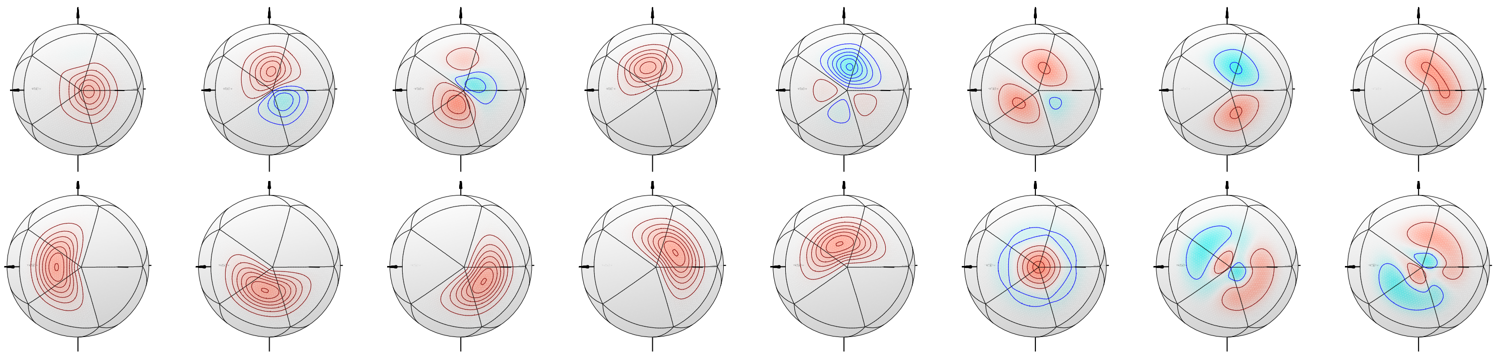
For the circle:

$$\mathcal{P}_r^d[T]|\mathbf{S}^1 \quad \supset \quad \mathcal{H}_r^d[T]|\mathbf{S}^2 \oplus \mathcal{H}_r^{d-1}[T]|\mathbf{S}^2.$$

ANS Finite Element Bases

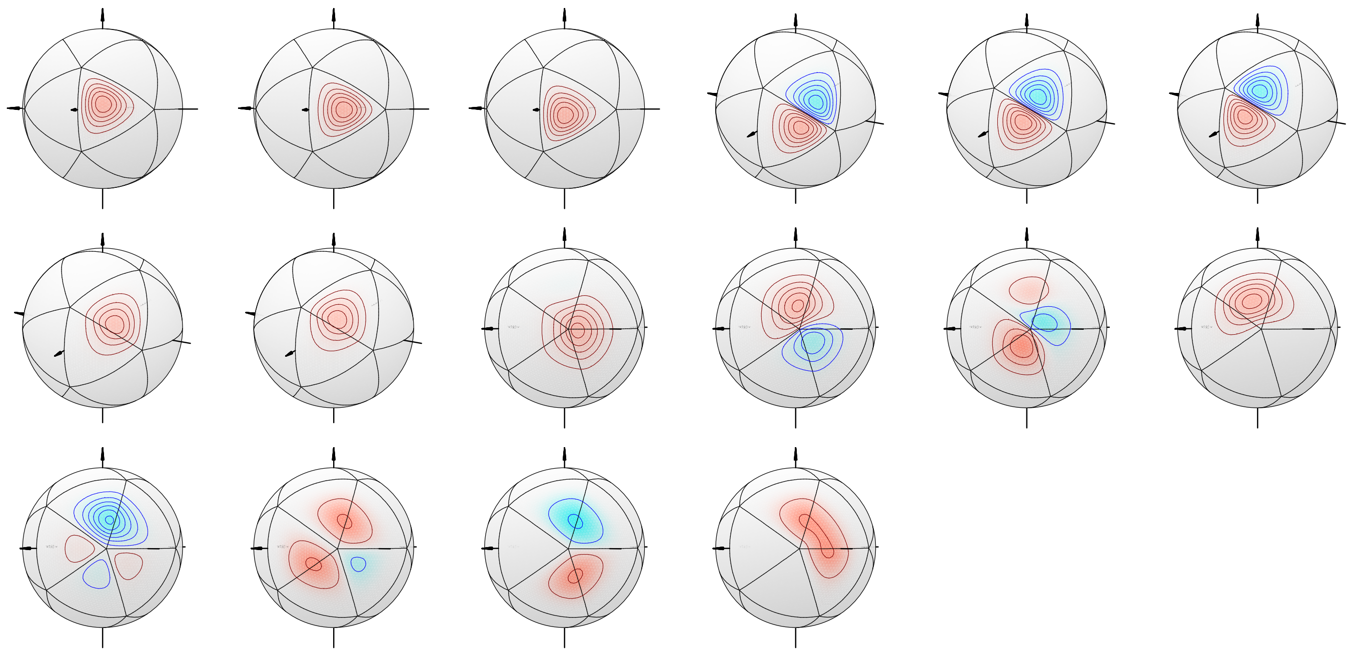
Two bases for each space, differ on vertex elements:

- Original construction (O): $g + 3$ “oranges”.
- New construction (N): g “boats” and 3 “oranges”.



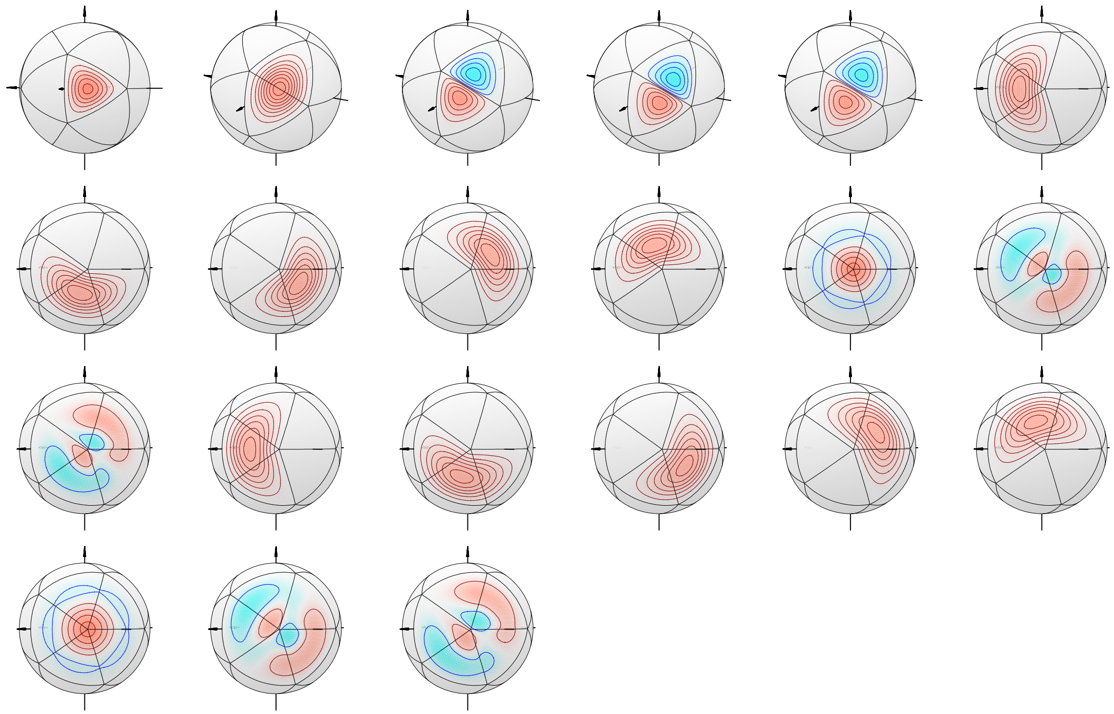
Original ANS basis

Original ANS basis for $\mathcal{H}_1^7$



Modified ANS basis

Modified ANS basis for $\mathcal{H}_1^6 \oplus \mathcal{H}_1^5$



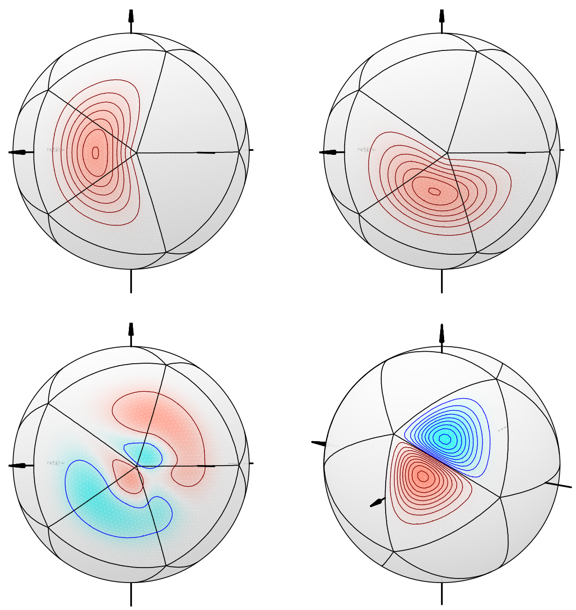
Least Squares Approximation

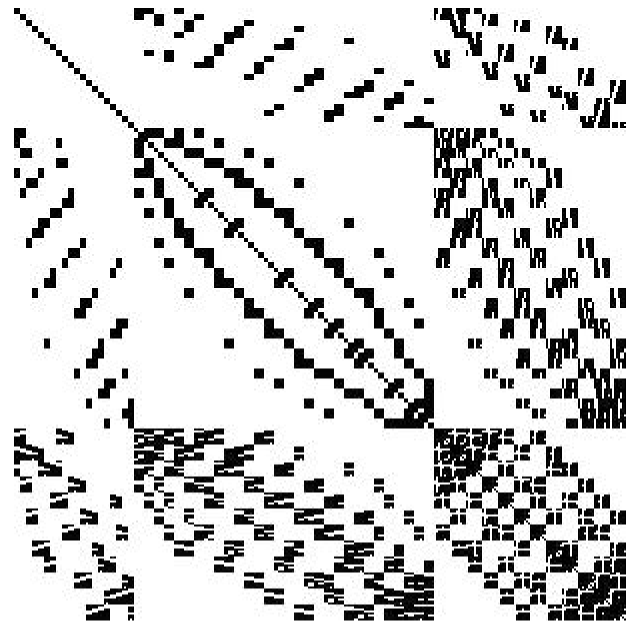
Least squares approximation problem
Solve $Ac = b$ where

$$\begin{aligned}
A_{i,j} &= \langle\, \phi_i \mid \phi_j \,\rangle \\
b_i &= \langle\, \phi_i \mid f \,\rangle \\
\phi_1, .. \, \phi_n &= \text{a function basis} \\
f &= \text{function to be approximated} \\
\langle \quad \mid \quad \rangle &= \text{a functional scalar product} \\
c_1, .. \, c_n &= \text{coefficients of approximation}
\end{aligned}$$

Matrices from Finite Elements

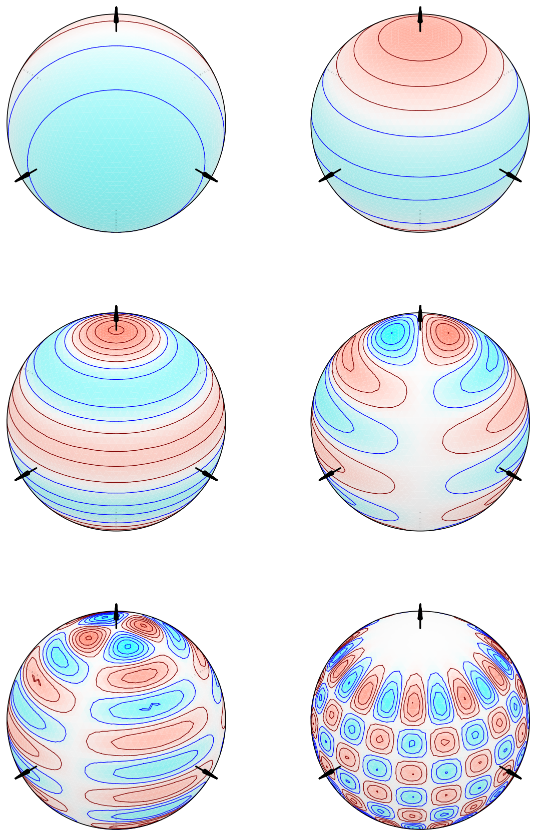
Finite-element bases give sparse matrices:

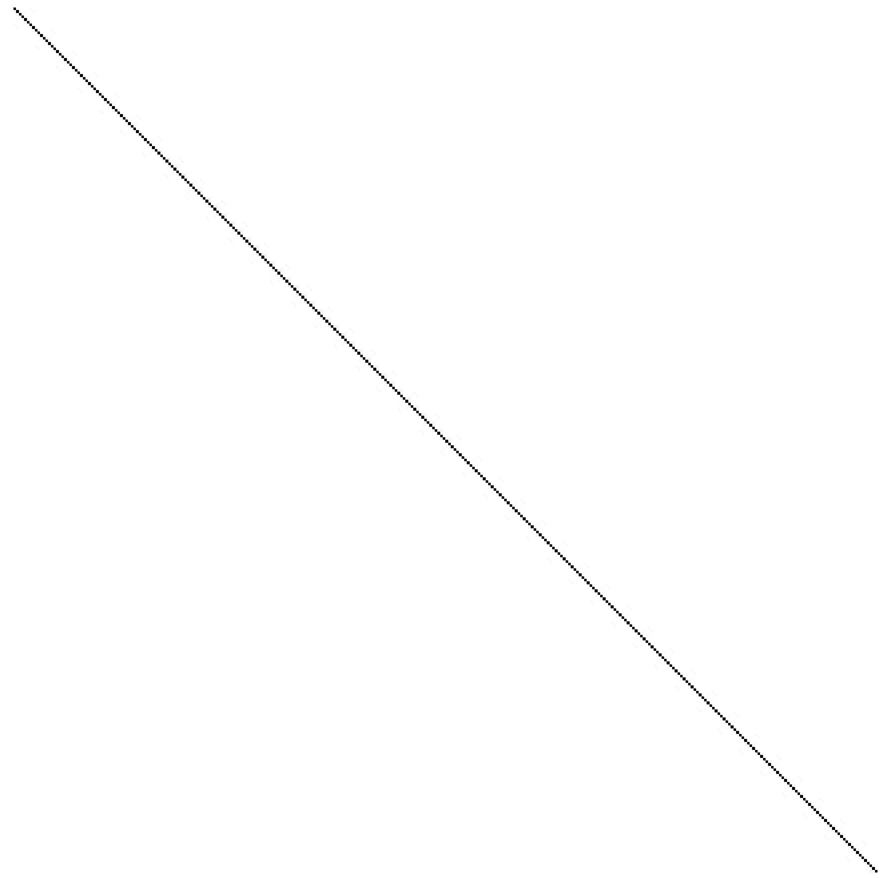




Orthogonal Bases

Orthogonal bases are much better:



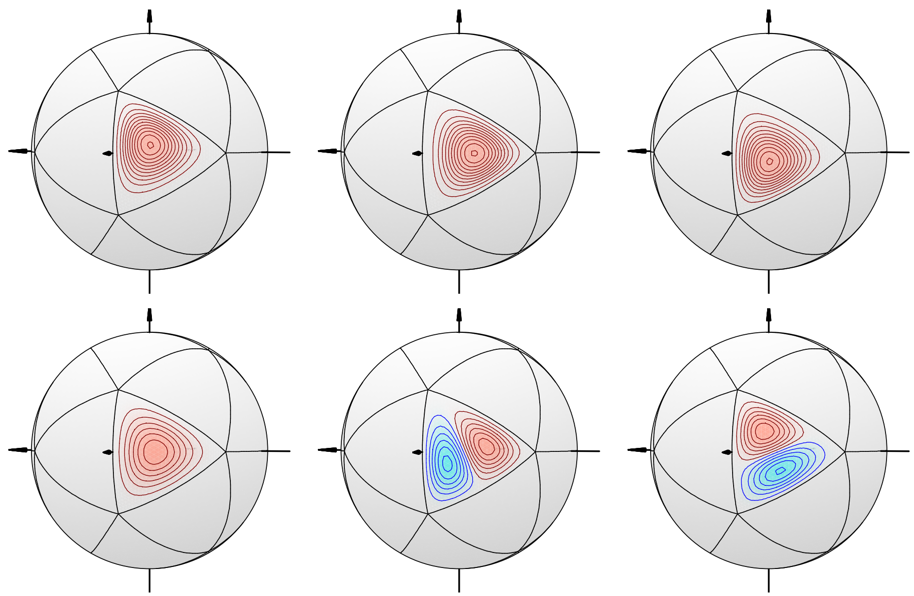


Semi-Orthogonalization

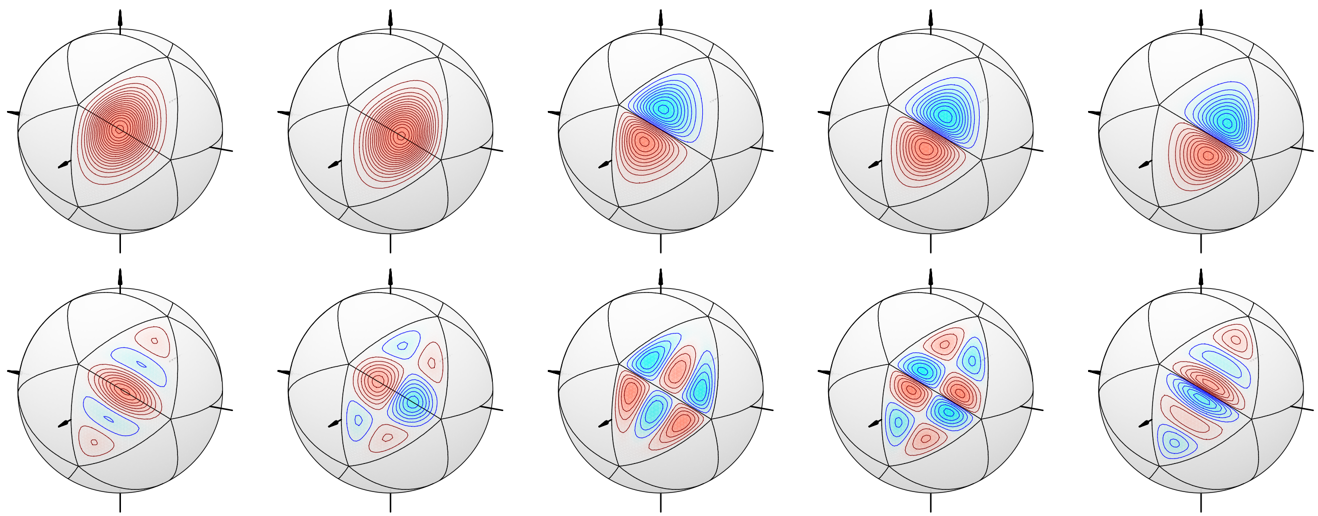
... but orthogonalization of a spline basis (e.g. Gram-Schmidt) destroys locality

Semi-orthogonalization:

Make the basis as orthogonal as possible
without increasing the element support



Semi-Orthogonalization (2)



Definitions and Notation

$$
\begin{aligned}
\Omega &= \text{the domain.} \\
T &= \text{the } \textit{mesh}\text{: a partition of } \Omega \text{ into } \textit{tiles } t_1, .. t_n. \\
\operatorname{supp}(f) &= \textit{support} \text{ of } f\text{: set of tiles of } T \text{ where } f \text{ is not zero.} \\
\Phi &= \text{a } \textit{basis}\text{: list of L.I. functions } \phi_1, .. \phi_m \text{ on } \Omega \\
\Phi_X &= \text{elements of } \Phi \text{ which have support } X.
\end{aligned}
$$

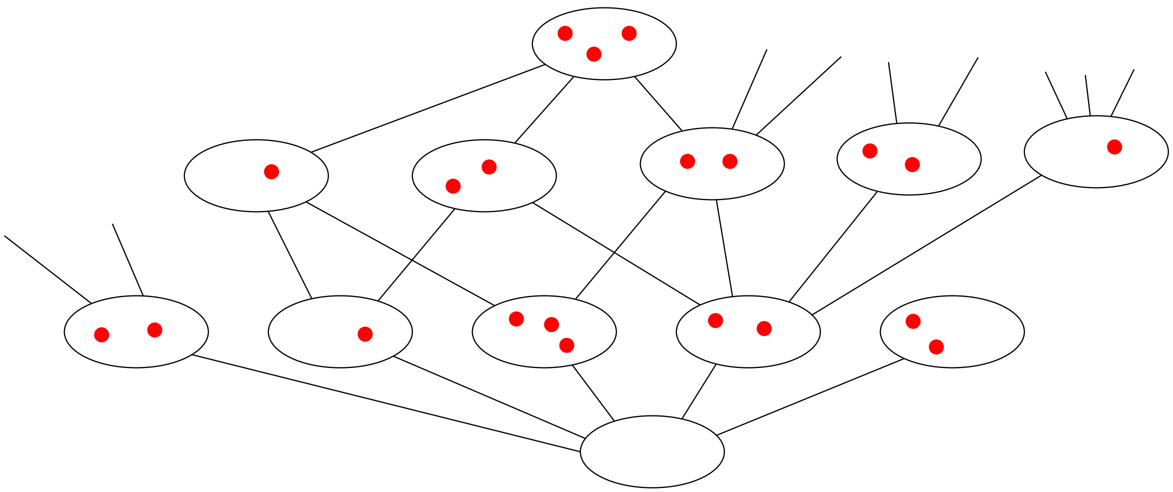
A basis Φ is *semi-orthogonal* iff
$\langle \phi_i \mid \phi_j \rangle = 0$ whenever $\operatorname{supp}(\phi_i) \subseteq \operatorname{supp}(\phi_j)$.

Support Structure of a Spline Basis

The map $X \to \Phi_X$ defines the *support structure* of Φ

Viewed as a graph $\mathcal{G}(\Phi)$

- Nodes $\mathcal{VG} = \{\, X : X \subseteq T \,\}$.
- Edges $\mathcal{EG}$: $X \to Y$ iff $X \subseteq Y$ and $|X| + 1 = |Y|$.
- Vertex X is decorated with the set Φ_X.



The useful part of $\mathcal{G}(\Phi)$ is small – $O(n) = O(m)$, not 2^m.

Semi-Orthogonalization Algorithm

- for each X in $\mathcal{VG}$, in increasing containment order
  - let $\Gamma_X \leftarrow \bigcup \{ \Phi_Y : Y \subset X \}$
  - for each element ϕ of Φ_X,
    - make ϕ orthogonal to Γ_X
  - make the elements of Φ_X orthogonal among themselves.

Cost of Algorithm

Typically $|X|$, $|\Phi_X|$, and $|\Gamma_X|$ are $O(1)$ for any X with non-empty Φ_X.

No step is executed more than m times.

Total time is $O(m)$ (but with a respectable constant!).

Since $\mathcal{VG}$ is processed bottom-up, Γ_X is semi-orthogonal.

If $\gamma_i \in \Gamma_X$ has $|\text{supp}(\gamma_i)| = 1$, we can assume that it is orthogonal to every other element in Γ_X.

When making $\Phi_X = (\gamma_1, .. \phi_k)$ orthogonal, use the eigenvectors of the matrix $A_X = \langle \gamma_i \mid \gamma_j \rangle$.

Example

Data used in test:

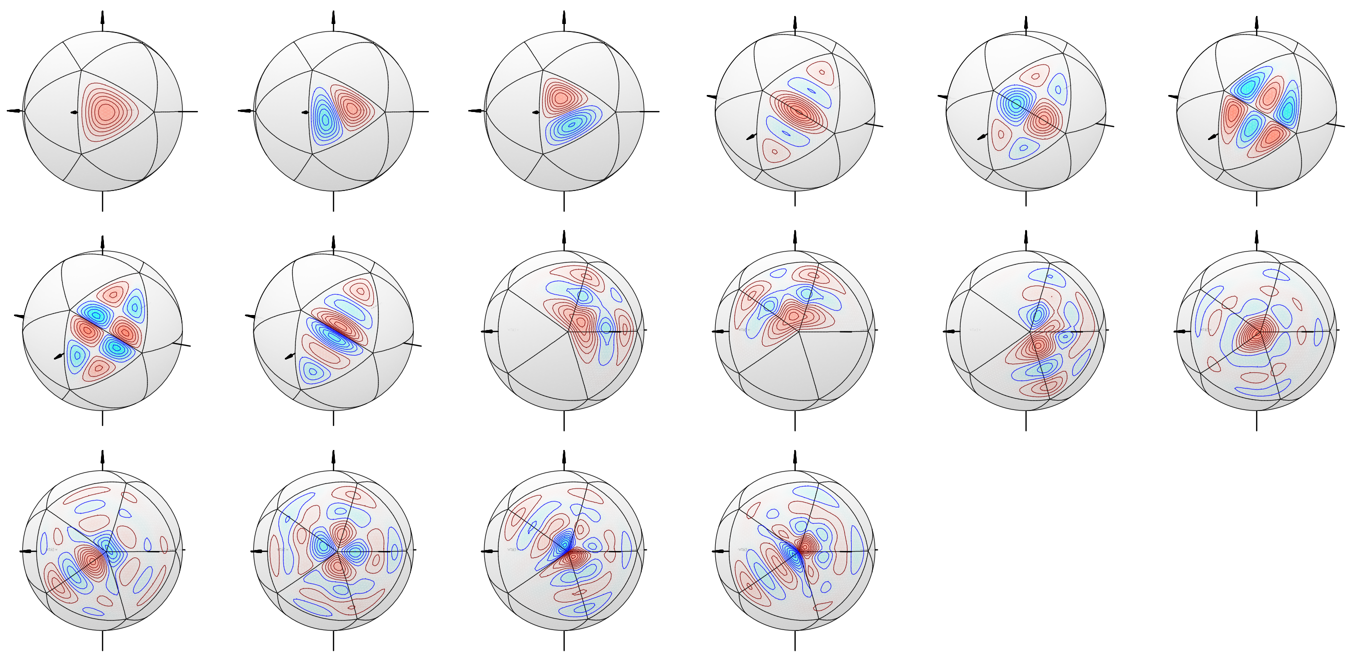
$$
\begin{aligned}
\Omega &= \text{sphere} \\
M &= \text{icosahedral triangulation of } \mathbf{S}^2 \\
\Phi &= \text{Alefeld-Neamtu-Schumaker (ANS) basis, } \mathsf{C}_1
\end{aligned}
$$

Two spaces:

- $\mathcal{H}_1^7$ - Homogeneous of degree 7 (dimension = 306)
- $\mathcal{H}_1^6 \oplus \mathcal{H}_1^5$ - Non-homogeneous of degree 6 (dimension = 332)

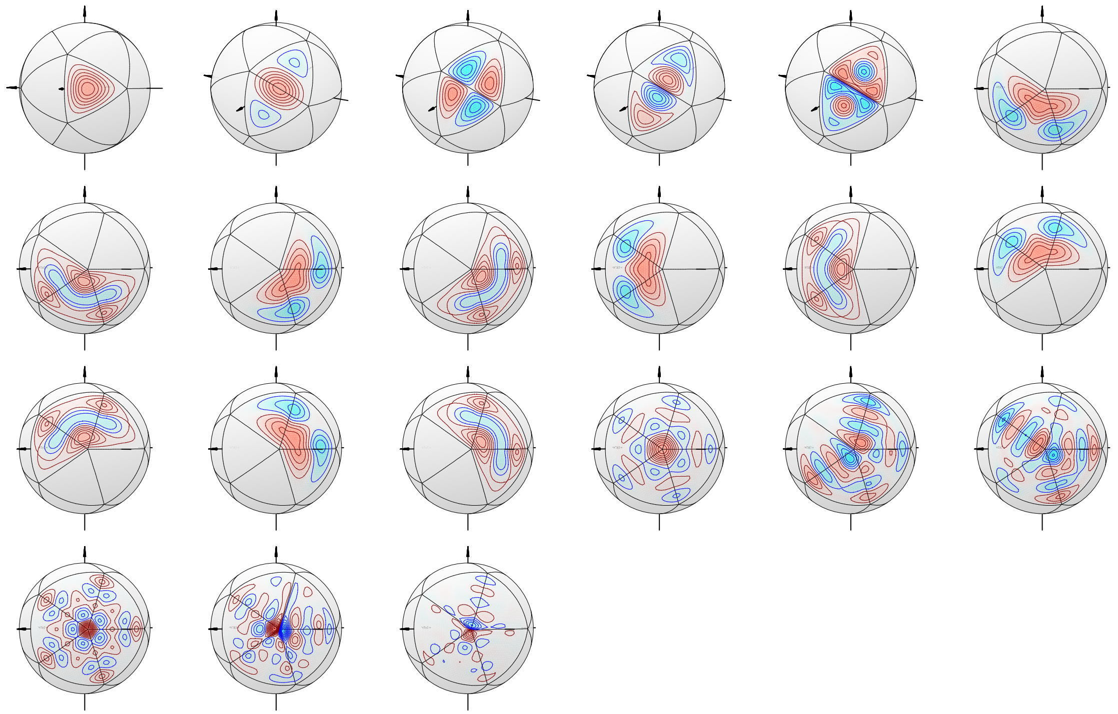
Original basis, semi-orthogonal

Semi-orthogonal ANS basis for $\mathcal{H}_1^7$



Modified basis, semi-orthogonal

Semi-orthogonal modified ANS basis for $\mathcal{H}_1^6 \oplus \mathcal{H}_1^5$



Comparison of bases

Non-zero elements and condition numbers

			plain		semi-ortho	
space	dim	basis	elems	cond	elems	cond
$\mathcal{H}_1^7$	306	O	21148	10^5	11316	10^2
$\mathcal{H}_1^7$	306	N	18226	10^5	10662	10^2
$\mathcal{H}_1^6 + \mathcal{H}_1^5$	332	O	35232	10^{13}	29708	10^8
$\mathcal{H}_1^6 + \mathcal{H}_1^5$	332	N	29440	10^{13}	26515	10^8

Static Linear PDE on the Sphere

Consider the differential equation

$$(\mathcal{D}f)(p) \;=\; R(p)$$

for all $p \in \Omega$ where

- f is a function defined on Ω, to be determined.
- $\mathcal{D}$ is a linear differential operator.
- R is a given function.

Example - Helmholtz equation

Helmholtz equation, $\Omega = \text{sphere}$

$$(\triangle f - cf)(p) = R(p)$$

where c is a constant,
$\triangle$ is the spherical Laplace-Beltrami operator, and
R is a given scalar field.

Aproximate Solution

Finite-element basis $\Phi = (\phi_0, .. \phi_{n-1})$.

Goal: compute

$$g = \sum_i c_i \phi_i$$

that approximates the solution f.

Galerkin's criterion:

$$\langle \mathcal{D}g - R \mid \phi_i \rangle = 0 \qquad \text{for } i = 0, .. n-1$$

where

$$\langle u \mid v \rangle = \int_\Omega u(p)v(p)\,dp.$$

Matrix Form

Galerkin's equations reduce to $Hc = b$ where

$$H_{ij} = \langle \mathcal{D}\phi_j - R \mid \phi_i \rangle$$

$$b_i = \langle R \mid \phi_i \rangle$$

For the Helmholtz equation

$$H_{ij} = \langle \mathring{\Delta}\, \phi_j \mid \phi i \rangle - c\langle \phi_j \mid \phi i \rangle$$

By Green's theorem, $\langle \mathring{\Delta}\, \phi_j \mid \phi i \rangle = \langle \mathring{\nabla}\, \phi_j \mid \mathring{\nabla}\, \phi i \rangle$
where $\mathring{\nabla}$ is the spherical gradient

$$(\mathring{\nabla}\, f)(p) = (\mathrm{grad}\, f)(p) - (p\langle \mathrm{grad}\, | f \rangle)p$$

Non-Linear Problems

More generally

$$(\mathcal{D}f)(p) = R(f(p), p)$$

for all $p \in \Omega$, where R may depend non-linearly on f.

- solve by iteration.
- remove the linear part of R (Newton).

Some Results

RHS	sol	rms error (#iters)	
		$\mathcal{H}^7$	$\mathcal{H}^6 \oplus \mathcal{H}^5$
`sqrx` $6.5x^2 - 2$	x^2	4.5×10^{-5}	1.0×10^{-12}
`cubx` $12.5x^3 - 6x$	x^3	9.0×10^{-13}	1.0×10^{-12}
`qrtx` $20.5x^4 - 12x^2$	x^4	4.4×10^{-5}	2.0×10^{-12}
`sepx` $56.5x^7 - 42x^5$	x^7	9.0×10^{-13}	6.9×10^{-5}
`octx` $72.5x^8 - 56x^6$	x^8	8.9×10^{-5}	8.6×10^{-5}
`cosx` $(1.5 - x^2)\cos x - 2x \sin x$	$\cos x$	8.1×10^{-5}	2.1×10^{-9}
`sinx` $(1.5 - x^2)\sin x + 2x \cos x$	$\sin x$	3.1×10^{-10}	1.3×10^{-8}
`expx` $e^x(x^2 + 2x - 0.5)$	e^x	1.1×10^{-4}	1.4×10^{-8}
`mcos` $(1.5 - x^2)(3\cos x + f)/4 - 2x \sin x$	$\cos x$	8.1×10^{-5} (25)	2.6×10^{-9} (40)
`mexp` $0.5(e^x + f)(x^2 + 2x - 0.5)$	e^x	1.1×10^{-4} (19)	1.4×10^{-8} (35)

Dynamic Problems

Time-dependent problem

$$(\mathcal{D}f)(p,t) = R(f(p,t),p,t)$$

where

$$\begin{aligned} p &= \text{point of } \Omega \\ t &= \text{time} \\ f &= \text{function to be determined} \\ \mathcal{D} &= \text{a linear differential operator} \\ R &= a\text{given function.} \end{aligned}$$

Example - Heat diffusion on a rotating sphere

$$(\frac{\partial}{\partial t} f)(p,t) - K\,(\triangle f)(p,t) + \omega\, V(p) \cdot (\nabla f)(p,t) + L\, f(p,t) = R(p,t)$$

where

$$\begin{aligned} K &= \text{heat diffusion coefficient} \\ \omega &= \text{angular speed} \\ V &= \text{velocity field for } \omega = 1\text{: } V(x,y,z) = (-y, x, 0) \\ L &= \text{coefficient of linear heat loss} \\ R &= \text{external heat input (may depend on } f(p,t)\text{)} \end{aligned}$$

Petrov-Galerkin Approach

Assume a basis of space-time elements

$$\Psi = (\psi_0, ..\, \psi_{N-1})$$

where each ψ_ℓ is a function of $\Omega \times \mathbf{R}$.

We look for an approximation h to f

$$h(p) = \sum_{\ell} c_\ell \psi_\ell(p, t)$$

where the c_ℓ are coefficients to be determined.

Petrov-Galerkin Approach (2)

We can use Petrov-Galerkin by
treating time as another space dimension:

$$\langle\, \mathcal{D}h - R \mid \theta_k \,\rangle = 0$$

for all k, where

$$\Theta = (\theta_0, .. \theta_{N-1})$$

is another basis of space-time elements (*gauge functions*).

We get the non-linear system

$$Mc = b$$

where

$$M_{k\ell} = \langle\, \mathcal{D}\psi_\ell \mid \theta_k \,\rangle \qquad b_k = \langle\, R \mid \theta_k \,\rangle$$

Separating Time and Space

We can partially separate the variables by using tensor-type bases for Φ and Θ. Let

$$\begin{aligned}\Phi &= (\phi_0, .. \phi_{n-1}) = \text{a basis over } \Omega \\ \Lambda &= (\phi_0, .. \phi_{m-1}) = \text{a basis over } \mathbf{R} \\ \Gamma &= (\gamma_0, .. \gamma_{m-1}) = \text{another basis over } \mathbf{R}\end{aligned}$$

Then use

$$\begin{aligned}\psi_k &= \phi_i \lambda_r \\ \theta_k &= \phi_i \gamma_r\end{aligned}$$

where$i = 0, .. n-1$, $r = 0, .. m-1$, and $k = in + r$.

Separating the Integrals

Integrals can be separated:

$$
\begin{aligned}
\langle\, \psi_\ell \mid \theta_k \,\rangle &= \langle\, \phi_j \mid \phi_i \,\rangle\langle\, \lambda_s \mid \gamma_r \,\rangle \\
\langle\, \mathring{\Delta}\, \psi_\ell \mid \theta_k \,\rangle &= \langle\, \mathring{\Delta}\, \phi_j \mid \phi_i \,\rangle\langle\, \lambda_s \mid \gamma_r \,\rangle \\
\langle\, V \cdot \mathring{\nabla}\, \psi_\ell \mid \theta_k \,\rangle &= \langle\, V \cdot \mathring{\nabla}\, \phi_j \mid \phi_i \,\rangle\langle\, \lambda_s \mid \gamma_r \,\rangle \\
\langle\, \frac{\partial}{\partial t}\psi_\ell \mid \theta_k \,\rangle &= \langle\, \phi_j \mid \phi_i \,\rangle\langle\, \frac{\partial}{\partial t}\lambda_s \mid \gamma_r \,\rangle
\end{aligned}
$$

Progressive Solution

Specifically, for the sphere we use

$$
\begin{aligned}
\Phi &= \text{the ANS spherical splines } \mathsf{C}_\mu \\
\Lambda &= \mathsf{C}_\mu \text{ spline pulses spanning 2 time steps} \\
\Gamma &= \text{spline pulses of degree} \leq \mu \text{ spanning one step}
\end{aligned}
$$

The time element λ_r is centered at epoch $\lfloor r/q \rfloor$ where $q = \mu + 1$.

Progressive Solution (2)

The system's matrix M then has the block structure

$$\begin{pmatrix} N_1 & N_0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & N_1 & N_0 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & N_1 & N_0 & 0 & \cdots \\ \vdots & 0 & \vdots & 0 & N_1 & N_0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where each block N_k, for k=0..1, is the a matrix of the scalar products of the differentiated basis elements in epoch $j - k$ against the gauge functions in epoch j.

Then at each step we solve

$$N_0 a_0 == d - N_1 a_1$$

where a_0 is the coefficients for the present epoch, and a_1 those of the previous epoch, and d is the corresponding segment of b.

An Example Result