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# Trees, Tight Extensions of Metric Spaces, and the Cohomological Dimension of Certain Groups: A Note on Combinatorial Properties of Metric Spaces

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The concept of tight extensions of a metric space is introduced, the existence of an essentially unique maximal tight extension $T_X$—the "tight span," being an abstract analogon of the convex hull—is established for any given metric space $X$ and its properties are studied. Applications with respect to (1) the existence of embeddings of a metric space into trees, (2) optimal graphs realizing a metric space, and (3) the cohomological dimension of groups with specific length functions are discussed.

*Contents.* 0. Introduction. 1. The tight span of a metric space. 2. Proofs of Theorems 1–4. 3. Optimal Networks. 4. Trees. 5. The combinatorial dimension of metric spaces. 6. Strongly discrete spaces and pseudo-convex polytopes. Appendix.

## 0. INTRODUCTION

Let $X$ be a metric space with $\#X \geqslant 2$ and let $D$: $X \times X \to \mathbb{R}$: $(x, y) \mapsto D(x, y) =: xy$ denote its distance map, so we have

(D1) $xy = yx \geqslant 0$,

(D2) $xy = 0 \Leftrightarrow x = y$, and

(D3) $xy + yz \geqslant xz$

for all $x, y, z \in X$. $X$ is defined to be a (metric) tree, if it satisfies the following two conditions:

(T1) For any $x, y \in X$ there exists a unique isometric embedding $\varphi = \varphi_{x,y}$ of the closed interval $[0, xy] \subseteq \mathbb{R}$ into $X$ such that $\varphi(0) = x$ and $\varphi(xy) = y$.

(T2) For any injective continuous map $\varphi$: $[0, 1] \hookrightarrow X$: $t \mapsto x_t$ of the unit interval $[0, 1] \subseteq \mathbb{R}$ into $X$ and any $t \in [0, 1]$ one has $x_0 x_t + x_t x_1 = x_0 x_1$.

Note that (T1) and (T2) together imply that for any two elements $x, y \in X$ in a tree there is —up to parametrization—only one injective continuous



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map $\varphi$: $[0, 1] \to X$: $t \mapsto x_t$ with $x_0 = x$ and $x_1 = y$, namely the map given by $x_t = \varphi_{x,y}(t \cdot xy)$ $(t \in [0, 1])$.

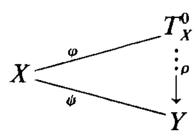
It has been shown (cf. [28, 3, 22, 16, 6]) that a metric space $X$ can be embedded isometrically into a tree if and only if $X$ satisfies the following condition:

(T) For any $x, y, v, w \in X$ the inequality $xy + vw > xv + yw$ implies $xy + vw = xw + yv$

or, equivalently,

(T′) For all $x, y, v, w \in X$ one has $xy + vw \leqslant \sup(xv + yw, xw + yv)$, (in which case $X$ will be called tree-like).

Moreover, if $X$ is tree-like, then $X$ determines a "smallest" tree $T_X^0$ in which it can be embedded uniquely up to isomorphism, i.e., for any tree-like $X$ there exists a tree $T_X^0$ and an isometric embedding $\varphi$: $X \hookrightarrow T_X^0$ such that for any other isometric embedding $\psi$: $X \hookrightarrow Y$ into a tree $Y$ there exists a unique isometric embedding $\rho$: $T_X^0 \to Y$ with $\rho \circ \varphi = \psi$:




There are several interesting applications of the construction $X \rightsquigarrow T_X^0$ which fall into the area of combinatorial group theory (cf. [15, 16, 2, 12]), and—maybe even more importantly—there are quite a few papers trying to approximate a given finite metric by tree-like metrics which fall into the area of mathematical taxonomy, i.e., which are concerned with the reconstruction of phylogenetic (or other) trees from distance matrices representing the (weighted) dissimilarity of present species (cf. [2, 5–8, 18, 19, 23–25, 27, 29, 35, 36]). While trying to understand the significance of the construction $X \rightsquigarrow T_X^0$ from a purely mathematical point of view it turned out that it can be extended to a construction, denoted by $X \rightsquigarrow T_X$, which is defined for arbitrary metric spaces and, in a way, mimics the convex hull construction defined for subsets of linear spaces. It is the purpose of this paper to introduce this rather natural construction and the quite elementary concepts related to it, as well as to discuss some of its properties and its applications.

To this end we define an extension $Y$ of a metric space $X$ to be a tight extension, if for any map $d$: $Y \times Y \to \mathbb{R}$ satisfying the conditions (D1) and (D3) above as well as the conditions $d(x_1, x_2) = x_1 x_2$ for all $x_1, x_2 \in X$ and $d(y_1, y_2) \leqslant y_1 y_2$ for all $y_1, y_2 \in Y$ one has necessarily $d(y_1, y_2) = y_1 y_2$ for all $y_1, y_2 \in Y$; thus, for example, the completion $\bar{X}$ of $X$ is necessarily a tight extension of $X$ for any metric space $X$. More generally, we have

THEOREM 1. *An extension Y of a metric space X is tight if and only if*

$$y_1 y_2 = \sup(x_1 x_2 - x_1 y_1 - y_2 x_2 \mid x_1, x_2 \in X)$$

*holds for all* $y_1, y_2 \in Y$.

A metric space $X$ is defined to be fully spread, if it has no proper tight extension—so a fully spread metric space $X$ is necessarily complete. Fully spread spaces are characterized by

THEOREM 2. *For a metric space X the following conditions are equivalent*:

(i) *For any map* $f\colon X \to \mathbb{R}$ *satisfying* $f(x) + f(y) \geqslant xy$ *for all* $x, y \in X$ *there exists some* $x \in X$ *with* $f(y) \geqslant xy$ *for all* $y \in X$.

(ii) *For any subspace* $Y \subseteq X$ *and any map* $f\colon Y \to \mathbb{R}$ *satisfying* $f(x) + f(x) \geqslant xy$ *for all* $x, y \in Y$ *there exists some* $x \in X$ *with* $f(y) \geqslant xy$ *for all* $y \in Y$.

(iii) *For any* $f\colon X \to \mathbb{R}$ *satisfying* $f(x) = \sup(xy - f(y) \mid y \in X)$ *for all* $x \in X$ *there exists some* $x \in X$ *with* $f(x) = 0$.

(iv) *For any subspace* $Y \subseteq X$ *and any* $f\colon Y \to \mathbb{R}$ *satisfying* $f(y) = \sup(yz - f(z) \mid z \in Y)$ *for all* $y \in Y$ *there exists some* $x \in X$ *with* $f(y) = xy$ *for all* $y \in Y$.

(v) *X is fully spread.*

Moreover, if $X$ is compact, then it is fully spread if and only if for any finite subset $Y \subseteq X$ and any map $f\colon Y \to \mathbb{R}$ with $f(y) + f(z) \geqslant yz$ for all $y, z \in Y$ there exists some $x \in X$ with $xy \leqslant f(y)$ for all $y \in Y$ if and only if for any finite subset $Y \subseteq X$ and any map $f\colon Y \to \mathbb{R}$ with $f(y) = \max(zy - f(z) \mid z \in Y)$ for all $y \in Y$ there exists some $x \in X$ with $xy = f(y)$ for all $y \in Y$.

Finally, if $X$ is fully spread, then it is contractible; more precisely, for any $x \in X$ there exists a homotopy $[0, 1] \times X \to X$: $(t, y) \mapsto H_t(y)$ satisfying $yH_t(y) = t \cdot xy$ and $xH_t(y) = (1 - t) \cdot xy$ for all $y \in X$.

That for any metric space $X$ there exists an essentially unique maximal tight extension, the "tight span" of $X$, follows from

THEOREM 3. *For a metric space X let* $T_X$ *denote the set of all* $f\colon X \to \mathbb{R}$ *satisfying*

$$f(x) = \sup(xy - f(y) \mid y \in X)$$

*for all* $x \in X$. *For any* $x \in X$ *let* $h_x$ *denote the map* $h_x\colon X \to \mathbb{R}\colon y \mapsto xy$.

*For any two maps $f$, $g$: $X \to \mathbb{R}$ define $\|f, g\|$ by $\|f, g\| =: \sup(|f(x) - g(x)| \mid x \in X) \in \mathbb{R} \cup \{\infty\}$. Then the following hold:*

(i) *$h_x \in T_X$ for all $x \in X$.*

(ii) *$\|h_x, f\| = f(x)$ for all $x \in X$ and all $f \in T_X$.*

(iii) *$\|g, f\| = \sup(g(x) - f(x) \mid x \in X) = \sup(xy - g(y) - f(x) \mid x, y \in X) \leqslant g(x) + f(x)$ for all $f$, $g \in T_X$ and $x \in X$.*

(iv) *For all $f \in T_X$ and all $x, y \in X$ one has $|f(x) - f(y)| \leqslant xy$.*

(v) *For $f$, $g : X \to \mathbb{R}$ write $f \leqslant g$ if $f(x) \leqslant g(x)$ for all $x \in X$. Then $T_X$ consists of the set of minimal elements in $P_X = \{g: X \to \mathbb{R} \mid g(x) + g(y) \geqslant xy$ for all $x, y \in X\}$* (w.r.t. $\leqslant$) *and for any $g \in P_X$ there exists some $f \in T_X$ with $f \leqslant g$.*

(vi) *For any extension $Y$ of $X$ there exists an isometry $\varphi$: $T_X \hookrightarrow T_Y$ with $\varphi(f)|_X = f$ for all $f \in T_X$.*

(vii) *If $Y$ is a tight extension of $X$ and $f \in T_Y$, then $f|_X$ is contained in $T_X$ and the map*

$$T_Y \to T_X : f \mapsto f|_X$$

*is a bijection, satisfying $\|f, g\| = \|f|_X, g|_X\|$ for all $f, g \in T_Y$, i.e., $T_Y \to T_X$: $f \mapsto f|_X$ is an isomorphism.*

Altogether, it follows that $T_X$ is a metric space with respect to the map $T_X \times T_X \to \mathbb{R}$: $(f, g) \mapsto \|f, g\|$, that $X \hookrightarrow T_X$: $x \mapsto h_x$ is an isometric embedding, that $T_X$ considered as an extension of $X$ is tight, that $T_X$ is compact if and only if the completion $\bar{X}$ of $X$ is compact and that an extension $Y$ of $X$ is tight, if and only if the map $Y \to T_X$: $y \mapsto h_y|_X$ is an isometric embedding, in which case it is the only isometric embedding $\psi$: $Y \to T_X$ satisfying $\psi(x) = h_x$ for all $x \in X$ and it extends to an isomorphism $T_Y \to T_X : f \mapsto f|_X$.

In other words, $T_X$ is the "universal tight extension" or the "tight span" of $X$, $T_X$ is fully spread for all metric spaces $X$ and a space $X$ itself is fully spread if and only if the embedding $X \hookrightarrow T_X$: $x \mapsto h_x$ is surjective or, equivalently, an isomorphism.

In case $X$ is compact then—analogously to the theory of compact convex spaces—one can always find a uniquely determined smallest compact subset of $X$, the "frame of $X$" denoted by $F_X$ such that $X$ is a tight extension of $F_X$. This is stated in detail in

THEOREM 4. *Let $X$ be a compact metric space and let $Y$ be a closed subspace of $X$. Then the following two conditions are equivalent:*

(i) *$X$ is a tight extension of $Y$,*

(ii) *Y contains the set* $F_X$ *of all* $y \in X$ *for which there exists some* $x \in X$ *with* $xy + yz > xz$ *for all* $z \in X \backslash \{y\}$.

*In particular, for any* $x_1, x_2 \in X$ *there exist* $y_1, y_2 \in F_X$ *such that*

$$y_1 y_2 = y_1 x_1 + x_1 x_2 + x_2 y_2$$

*and for any* $x \in X$ *and* $y \in F_X$ *there is some* $z \in F_X$ *with* $zy = zx + xy$.

If $X$ is a finite metric space then $T_X$ is closely related to the problem of constructing "optimal realizations of $X$ by networks," i.e., of constructing systems $\Gamma = (V, \mathscr{E}, l)$ with $V$ being a set containing $X$ and representing the vertices of $\Gamma$, $\mathscr{E}$ being a subset of $\mathscr{P}_2(V) = \{e \subseteq V \mid \#e = 2\}$, representing the edges of $\Gamma$, with $\operatorname{supp} \mathscr{E} =: \bigcup_{e \in \mathscr{E}} e = V$ and $l: \mathscr{E} \to \mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$ being a "length function" such that for any $x, y \in X$ with $x \neq y$ one has $xy = \inf(l(\{v_0, v_1\}) + l(\{v_1, v_2\}) + \cdots + l(\{v_{n-1}, v_n\}) \mid n \in \mathbb{N};\ v_0 = x,\ v_1, ..., v_{n-1}, v_n = y \in V;\ \{v_0, v_1\}, ..., \{v_{n-1}, v_n\} \in \mathscr{E})$ and such that $\|\Gamma\| =: \sum_{e \in \mathscr{E}} l(e)$ is minimal with respect to these properties (cf. [10, 22, 13, 31, 14]). If $\Gamma = (V, \mathscr{E}, l)$ satisfies all of the above conditions except perhaps the minimality condition concerning $\|\Gamma\|$, then $\Gamma$ will be called a realization of $X$.

Concerning optimal realizations of finite metric spaces by networks we can show

THEOREM 5. *If* $X$ *is a finite metric space and if* $\Gamma = (V, \mathscr{E}, l)$ *is an optimal realization of* $X$, *then there exists a map* $\psi: V \to T_X$ *with* $\psi(x) = h_x$ *for all* $x \in X$ *and* $\|\psi(v_1), \psi(v_2)\| = l(\{v_1, v_2\})$ *for all* $v_1, v_2 \in V$ *with* $\{v_1, v_2\} \in \mathscr{E}$.

In view of Theorem 5 the following simple observation can be considered as a generalization of a result concerning optimal realizations which has been proved by Imrich and Stotzkii (cf. [14]).

THEOREM 6. *If* $X$ *is a metric space and if there exists a nontrivial partition* $X = Y \cup Z$ *and a map* $f: X \to \mathbb{R}$ *satisfying* $f(x) + f(y) \geqslant xy$ *for all* $x, y \in X$ *as well as* $f(y) + f(z) = yz$ *for all* $y \in Y$ *and* $z \in Z$, *then* $f \in T_X$ *and* $T_X \backslash \{f\}$ *is the disjoint union of the two open subsets* $\mathscr{O}_Y = \{g \in T_X \mid$ *there exists some* $y \in Y$ *with* $g(y) < f(y)\}$ *and* $\mathscr{O}_Z = \{g \in T_X \mid$ *there exists some* $z \in Z$ *with* $g(z) < f(z)\}$.

*In particular, for any isometry* $\varphi: [0, r] \to T_X$ *with* $g_1 = \varphi(0) \in \mathscr{O}_Y$ *and* $g_2 = \varphi(r) \in \mathscr{O}_Z$ *one has necessarily* $f \in \varphi([0, r])$ *and thus one has* $\|g_1, g_2\| = \|g_1, f\| + \|f, g_2\|$ *for any* $g_1 \in \mathscr{O}_Y$ *and* $g_2 \in \mathscr{O}_Z$.

It should be noted that so far no efficient construction of all or at least of one optimal realization of a given finite metric space seems to be known and that there are finite metric spaces $X$ with $\#X = 5$ which have two essentially nonisomorphic optimal realizations (see the Appendix). Thus the following observation may be of interest: for a finite metric space $X$ we define hereditarily optimal realizations $(V, \mathscr{E}, l)$ of $X$ by induction with respect to $\#X$: if $\#X \leqslant 2$, then any optimal realization of $X$ is defined to be hereditarily optimal. If $\#X = k$ and if hereditarily optimal realizations of $Y$ have been defined already for all metric space $Y$ with $\#Y < k$, then a realization $\Gamma = (V, \mathscr{E}, l)$ of $X \subseteq V$ is defined to be hereditarily optimal if for any $Y \subsetneqq X$ there is some $\mathscr{E}' \subseteq \mathscr{E}$ such that $Y \subseteq V' =: \operatorname{supp} \mathscr{E}'$ and $\Gamma' =: (V', \mathscr{E}', l|_{\mathscr{E}'})$ is a hereditarily optimal realization of $Y$, if $V = \operatorname{supp} \mathscr{E}$ and if $\|\Gamma\|$ is minimal with respect to these properties.

Concerning hereditarily optimal realizations we can show

THEOREM 7. *For a finite metric space $X$ let $V_x$ denote the set of all $f \in T_X$ for which the symmetric relation*

$$\mathscr{K}_f =: \{(x, y) \in X \times X \mid f(x) + f(y) = xy\} \subseteq X \times X$$

*is $X$-connected and nonbipartite (i.e., for which $\mathscr{K}_f$ satisfies $X \times X = \bigcup_{n \in \mathbb{N}} \mathscr{K}_f^{2n}$ with $\mathscr{K}^n$ denoting the n-fold relational power of $\mathscr{K}$), identify $X$ with $\{h_x \mid x \in V\} \subseteq V_X$, let $\mathscr{E}_X$ denote the set of all subsets $\{f, g\} \subseteq V_X$ with $f \neq g$ for which $\mathscr{K}_f \cap \mathscr{K}_g$ is $X$-connected (i.e., $X \times X = \bigcup_{n \in \mathbb{N}} (\mathscr{K}_f \cap \mathscr{K}_g)^n$) and let $l_X : \mathscr{E}_X \to \mathbb{R}$ be defined by $l_X(\{f, g\}) = \|f, g\|$ for all $\{f, g\} \in \mathscr{E}_X$. Then $\Gamma_X = (V_X, \mathscr{E}_X, l_X)$ is a hereditarily optimal realization of $X$ and any other hereditarily optimal realization $\Gamma = (V, \mathscr{E}, l)$ is essentially isomorphic to $\Gamma_X$, i.e., it becomes isomorphic to $\Gamma_X$ once vertices $v \in V \backslash X$ with $\deg_\Gamma v = \#\{e \in \mathscr{E} \mid v \in e\} = 2$ have been deleted one by one and the corresponding edges $e_1 = \{v, u_1\}$, $e_2 = \{v, u_2\} \in \mathscr{E}$ have been replaced by $\{u_1, u_2\}$—with $l(\{u_1, u_2\}) = l(\{u_1, v\}) + l(\{v, u_2\})$, of course.*

*Remark.* It was this generalization of the theorem of Simões-Pereira mentioned above which originally motivated the study of tight spans of metric spaces.

Finally, we have to relate the $T_X$-construction to the original problem of embedding a tree-like metric space $X$ into a tree. This is done in

THEOREM 8. *Let $X$ be a metric space. Then the following conditions are equivalent:*

(i) *$X$ is tree-like;*

(ii) *$T_X$ is a tree;*

(iii) *$T_X^0 = \{f \in T_X \mid X = \operatorname{supp} \mathscr{K}_f =: \bigcup_{(x,y) \in \mathscr{K}_f} \{x, y\}\}$ is a tree;*

(iv) *the small inductive dimension* ind $T_X$ *of $T_X$ is* 1;

(iv′) *any closed separable subspace of* $T_X$ *has topological dimension* $\leqslant 1$, *in particular,* $\dim T_Y \leqslant 1$ *for any finite* $Y \subseteq X$;

(v) *For any* $Y \subseteq X$ *with* $\# Y = 4$ *one has* $\dim T_Y = 1$;

(vi) $\mathscr{K}_f$ *is completely multipartite for any* $f \in T_X$, *i.e., for any* $f \in T_X$ *is* $X \times X \backslash \mathscr{K}_f$ *an equivalence relation on its support*;

(vii) *X can be embedded isometrically into a tree.*

*Moreover, in this case, any isometric embedding of X into a tree T extends uniquely to an isometric embedding of* $T_X^0$ *into T. And, finally, a metric space X is a tree if and only if it is tree-like and connected, in which case its completion* $\bar{X}$ *coincides with* $T_X$.

*Remark.* I conjecture that even for nonseparable tree-like spaces $X$ we have $\dim T_X = 1$, so that $X$ is tree-like if and only if $\dim T_X = 1$.

In this context we mention still another result which generalizes part of Theorem 8 to higher dimensions.

THEOREM 9. *Let X be a metric space. Then the topological dimension of* $T_Y$ *is smaller than some* $n \in \mathbb{N}$ *for all finite subspaces* $Y \subseteq X$ *if and only if for all* $x_1, x_{-1}, x_2, x_{-2}, \ldots, x_n, x_{-n} \in X$ *there exists some permutation* $\alpha$ *of* $\{\pm 1, \pm 2, \ldots, \pm n\} = I$ *with* $\alpha \neq -\mathrm{Id}_I$ *and* $\sum_{i \in I} x_i x_{-i} \leqslant \sum_{i \in I} x_i x_{\alpha(i)}$.

*Remark.* It seems reasonable to conjecture that the above conditions are in turn equivalent to $\dim T_X < n$. This holds at least if $xy \in \mathbb{N} = \{0, 1, \ldots\}$ for all $x, y \in X$ which in turn implies the following generalization of a theorem of Lyndon (cf. [17], see also [4, 11, 12, 15, 34, 37]):

THEOREM 10. *Let G be a group and let* $l\colon G \to \mathbb{Z}$ *be a "length function," i.e., a map satisfying*

(L1) $l(g) = l(g^{-1}) \geqslant 0$,

(L2) $l(g) = 0 \Leftrightarrow g = 1$,

(L3) $l(gh) \leqslant l(g) + l(h)$, *and*

(L4) $\#\{g^n \mid n \in \mathbb{Z}\} < \infty \Leftrightarrow \sup(l(g^n) \mid n \in \mathbb{Z}) < \infty$.

*If for any* $x_1, x_{-1}, \ldots, x_n, x_{-n} \in G$ *one can find some permutation* $\alpha$ *of* $I =: \{\pm 1, \ldots, \pm n\}$ *with* $\alpha \neq -\mathrm{Id}_I$ *and* $\sum_{i \in I} l(x_i x_{-i}^{-1}) \leqslant \sum_{i \in I} l(x_i x_{\alpha(i)}^{-1})$, *then any torsion-free subgroup of G must have cohomological dimension smaller than n.*

More precisely, if $G$ is torsion free, if $\mathscr{R}_{(G,l)}^{(k)}$ denotes the set of all $\mathscr{K} \subseteq G \times G$ with $G = \operatorname{supp} \mathscr{K}$ for which there exists some $f\colon G \to \mathbb{R}$ with $\mathscr{K} = \{(x, y) \in G \times G \mid f(x) + f(y) = l(x^{-1}y)\}$ and $f(x) + f(y) \geqslant l(x^{-1}y)$ for all $x, y \in G$ and for which $W_{\mathscr{K}} =: \{v \in \mathbb{R}^G \mid v(x) + v(y) = 0$ for all

$(x, y) \in \mathscr{K}\}$ has dimension $k$, if for any such $\mathscr{K} \in \mathscr{R}^{(k)}_{(G,l)}$ one denotes by $G^k_{\mathscr{K}}$ the set of all sequences $(x_1, ..., x_k) \in G$ for which the map $\varphi_{(x_1,...,x_k)}$: $W_{\mathscr{K}} \to \mathbb{R}^k$: $v \mapsto (v(x_1), ..., v(x_k))$ is an isomorphism, if for $(x_1, ..., x_k)$, $(y_1, ..., y_k) \in G^k_{\mathscr{K}}$ one defines $\operatorname{sgn}((x_1, ..., x_k), (y_1, ..., y_k)) =: \operatorname{sgn}(\det(\varphi_{(x_1,...,x_k)} \circ \varphi^{-1}_{(y_1,...,y_k)}))$ and if one denotes by $C_k = C_k(G, l)$ the free abelian group, generated by all systems $(\mathscr{K}; x_1, ..., x_k)$ with $\mathscr{K} \in \mathscr{R}^{(k)}_{(G,l)}$ and $(x_1, ..., x_k) \in G^k_{\mathscr{K}}$ modulo the relations $(\mathscr{K}; x_1, ..., x_k) - \operatorname{sgn}((x_1, ..., x_k), (y_1, ..., y_k)) \cdot (\mathscr{K}; y_1, ..., y_k)$, then $0 \leftarrow \mathbb{Z} \leftarrow^d C_0(G, l) \leftarrow^\partial C_1(G, l) \leftarrow \cdots \leftarrow C_{n-1}(G, l) \leftarrow 0$ is a free resolution of the trivial $G$-module $\mathbb{Z}$ if $G$ acts on $C_k(G, l)$ by $g \cdot (\mathscr{K}; x_1, ..., x_k) = (g\mathscr{K}; gx_1, ..., gx_k)$ with $g\mathscr{K} =: \{(gx, gy) \mid (x, y) \in \mathscr{K}\}$, $d$: $C_0(G, l) \to \mathbb{Z}$ is defined by $d(\mathscr{K}) = 1$ and $\partial$: $C_{k+1}(G, l) \to C_k(G, l)$ is defined in the following way: if $\mathscr{L} \in \mathscr{R}^{k+1}_{(G,l)}$ and $(x_1, ..., x_{k+1}) \in G^{k+1}_{\mathscr{L}}$, then there are only finitely many $\mathscr{K} \in \mathscr{R}^{(k)}_{(G,l)}$ with $\mathscr{L} \subseteq \mathscr{K}$, for each such $\mathscr{K}$ there exists a minimal $i = i_{\mathscr{K}} \in \{1, ..., k+1\}$ such that $(x_1, ..., \hat{x}_i, ..., x_{k+1}) \in G^k_{\mathscr{K}}$, and some $v = v_{\mathscr{K}} \in W_{\mathscr{L}} \backslash W_{\mathscr{K}}$ with $v(x) + v(y) \geqslant 0$ for all $(x, y) \in \mathscr{K}$. Now define

$$\partial(\mathscr{L}; x_1, ..., x_{k+1}) =: \sum_{\mathscr{K} \in \mathscr{R}^k_{(G,l)}, \mathscr{L} \subseteq \mathscr{K}} \operatorname{sgn}(v_{\mathscr{K}}(x_{i_{\mathscr{K}}})) \cdot (-1)^{i_{\mathscr{K}}} \cdot (\mathscr{K}; x_1, ..., \hat{x}_{i_{\mathscr{K}}}, ..., x_{k+1}).$$

Another free resolution $0 \leftarrow \mathbb{Z} \leftarrow^d B_0(G, l) \leftarrow^\partial B_1(G, l) \leftarrow \cdots \leftarrow B_{n-1}(G, l) \leftarrow 0$ of the trivial $G$-module $\mathbb{Z}$ is obtained if one defines $B_k(G, l)$ to be the free abelian group generated by all sequences $(\mathscr{K}_0, \mathscr{K}_1, ..., \mathscr{K}_k)$ with $\mathscr{K}_0, \mathscr{K}_1, ..., \mathscr{K}_k \in \mathscr{R}_{(G,l)} =: \bigcup_{i \in \mathbb{N}} \mathscr{R}^{(i)}_{(G,l)}$ *and* $\mathscr{K}_0 \supsetneqq \mathscr{K}_1 \supsetneqq \cdots \supsetneqq \mathscr{K}_k$, $d$: $B_0(G, l) \to \mathbb{Z}$ by $d(\mathscr{K}) = 1$ and $\partial(\mathscr{K}_0, ..., \mathscr{K}_{k+1}) = \sum_{i=0}^{k+1} (-1)^i (\mathscr{K}_0, ..., \hat{\mathscr{K}}_i, ..., \mathscr{K}_k)$.

Note that the second resolution has much "larger" chain groups $B_k(G, l)$ though it has simpler boundary maps.

*Remark.* An example of a group with such a length function—though one, for which the conclusion of Theorem 10 is trivial—is $G = \mathbb{Z}^k$ with $l$: $\mathbb{Z}^k \to \mathbb{Z}$ given by

$$l((n_1, ..., n_k)) = \sup(|n_1|, ..., |n_k|).$$

In subsequent papers I will discuss some further minimality properties of $T_X$ as well as some results concerning the existence and the homogeneity of "universal $k$-trees" ($k \in \mathbb{N}$), i.e., of trees $T$ such that for any $x \in T$ the complement $T \backslash \{x\}$ consists of precisely $k$ connected components and which are complete as metric spaces.

The organization of the paper is as follows: in Section 1 the general theory of tight extensions is developed, in Section 2 the Theorems 1–4 are being deduced, in Section 3 the theory of optimal and hereditarily optimal networks is developed (with some remarks banned into an Appendix to this paper) and Theorems 5–7 are proved, Section 4 deals with the theory of trees (Theorem 8), Section 5 develops the concept of the combinatorial dimension of metric spaces (Theorem 9), and Section 6 finally builds up the techniques from which Theorem 10 is deduced.

*Note added in proof.* (1) As I learned in the meantime, the construction $X \rightsquigarrow T_X$ has already been studied by J. R. Isbell (*Comment. Math. Helv.* **39** (1964–65), 65–74), where results similar to those of Sections 1 and 2 are derived. (2) Results, which are closely connected to the results derived in the Appendix, have also be obtained by Imrich and Simoes–Pereira and will be published in *Journal of Combinatorial Theory, Series B.*

## 1. The Tight Span of a Metric Space

In this section we collect a number of simple facts which will be useful for the proofs of the above stated theorems later on.

**(1.1)** For a metric space $X$ with its distance map $D: X \times X \to \mathbb{R}: (x, y) \mapsto D(x, y) = xy$ let

$$P_X =: \{f: X \to \mathbb{R} \mid f(x) + f(y) \geqslant xy \text{ for all } x, y \in X\}$$

and

$$T_X =: \{f: X \to \mathbb{R} \mid f(x) = \sup(xy - f(y) \mid y \in X) \text{ for all } x \in X\}.$$

Note that $f(x) + f(x) \geqslant xx = 0$, i.e., $f(x) \geqslant 0$ for all $f \in P_X$ and $x \in X$ and that a map $f: X \to \mathbb{R}$ with $f(x) = 0$ for some $x \in X$ is in $P_X$ if and only if $f(y) = f(y) + f(x) \geqslant yx$ for all $y \in X$.

**(1.2)** Note also that for any $f \in P_X$ there exists a unique maximal subset $Y \subseteq X$ with $f|_Y \in T_Y$ (which may be empty) since $f \in P_X$, $f|_{Y_\alpha} \in T_{Y_\alpha}$ for a family $\{Y_\alpha \subseteq X \mid \alpha \in A\}$ of subsets of $X$ and $Y = \bigcup_{\alpha \in A} Y_\alpha$ implies $f(y) \geqslant \sup(yz - f(z) \mid z \in X) \geqslant \sup(yz - f(z) \mid z \in Y) \geqslant \sup(yz - f(z) \mid z \in Y_\alpha) = f(y)$ for all $\alpha \in A$ and $y \in Y_\alpha$ and thus $f(y) = \sup(yz - f(z) \mid z \in Y)$ for all $y \in Y$, i.e., $f|_Y \in T_Y$.

For any $Z \subseteq X$ let $P_X^Z =: \{f \in P_X \mid f|_{X \setminus Z} \in T_{X \setminus Z}\}$ and let $T_X^Z =: T_X \cap P_X^Z$. In case $Z = \{x\}$ write $P_X^x$ and $T_X^x$ instead of $P_X^{\{x\}}$ and $T_X^{\{x\}}$, respectively.

**(1.3)** It follows also directly from the definitions that $T_X$ consists of all $f \in P_X$ which are "minimal" in $P_X$, i.e., for which no $g \in P_X$ with $g \lneqq f$ exists. Since on the one hand, $g \leqslant f \in T_X$ and $g \in P_X$ implies $f(x) = \sup(xy - f(y) \mid y \in X) \leqslant \sup(xy - g(y) \mid y \in X) \leqslant g(x)$ and thus $f = g$. Whereas on the other hand, $f(x) > \sup(xy - f(y) \mid y \in X)$ for some $f \in P_X$ and some $x \in X$ allows us to introduce the map

$$p_x(f): X \to \mathbb{R}: z \mapsto \begin{cases} f(z) & \text{if } z \neq x, \\ \sup(0, xy - f(y) \mid y \in X) & \text{if } z = x, \end{cases}$$

which is obviously also in $P_X$ and satisfies $p_x(f)(z) \leqslant f(z)$ for all $z \in X$. But $p_x(f) \neq f$, since $p_x(f)(x)$ is either 0 (whereas $f(x) > \sup(xy - f(y) \mid y \in X)$ implies $f(x) > xx - f(x)$ and thus $f(x) > 0$) or $p_x(f)$ equals $\sup(xy - f(y) \mid y \in X)$ which is also supposed to be smaller than $f(x)$. Thus, using Zorn's lemma, one sees easily that for any $f \in P_X$ there exists some $g \in T_X$ with $g \leqslant f$.

Note for further use that the map $p_x\colon P_X \to P_X$ satisfies $p_x(f) \leqslant f$ by its very construction and $\|p_x(f), p_x(g)\| \leqslant \|f, g\|$ for all $f, g \in P_X$, since $p_x(f)(x) = \sup(0, xy - f(y) \mid y \in X) = \sup(0, (xy - g(y) + (g(y) - f(y)) \mid y \in X) \leqslant \sup(0, xy - g(y) \mid y \in X) + \sup(0, g(y) - f(y) \mid y \in X) \leqslant p_x(g)(x) + \|f, g\|$ and, just as well, $p_x(g)(x) \leqslant p_x(f)(x) + \|f, g\|$ which implies $\|p_x(f), p_x(g)\| = \sup(|p_x(f)(y) - p_x(g)(y)| \mid y \in X) \leqslant \sup(\|f, g\|, |f(y) - g(y)| \mid y \in X \backslash \{x\}) = \|f, g\|$.

**(1.4)** In particular, a map $f \in P_X$ with $f(x) = 0$ for some $x \in X$ and thus $f(y) \geqslant yx$ for all $y \in X$ is in $T_X$ if and only if $f$ equals

$$h_x : X \to \mathbb{R} : y \mapsto yx.$$

Thus $f \in T_X$ and $x \in X \neq \{x\}$ implies $f(x) = \sup(xy - f(y) \mid y \in X \backslash \{x\})$, since otherwise $f(x) = xx - f(x) > xy - f(y)$ for all $y \in X \backslash \{x\}$, i.e., $f(x) = 0$ and $f(y) > xy$ for all $y \in X \backslash \{x\} \neq \varnothing$, in contradiction to the above remark. In particular, $f, g \in T_X$ and $f(y) = g(y)$ for all $y \in X \backslash \{x\}$ implies $f = g$.

**(1.5)** Next we show that for any $f \in T_X$ its distance $\|h_x, f\| = \sup(|h_x(z) - f(z)| \mid z \in X)$ to $h_x$ equals $f(x)$: from $f(z) = \sup(zy - f(y) \mid y \in X) \leqslant \sup(zx + xy - f(y) \mid y \in X) = zx + \sup(xy - f(y) \mid y \in X) = h_x(z) + f(x)$ we get $f(z) - h_x(z) \leqslant f(x)$ with equality holding true for $z = x$, whereas $h_x(z) = xz \leqslant f(x) + f(z)$ implies $h_x(z) - f(z) \leqslant f(x)$, so altogether we have indeed $\sup(|h_x(z) - f(z)| \mid z \in X) = f(x)$.

**(1.6)** This formula has some interesting applications: at first it shows that the map $X \to T_X$: $x \mapsto h_x$ is an isometry, since $\|h_x, h_y\| = h_y(x) = xy$ for all $x, y \in X$, it shows also that $f, g \in T_X$ implies $\|f, g\| \leqslant \|f, h_x\| + \|h_x, g\| = f(x) + g(x)$ for all $x \in X$ and thus $\|f, g\| < \infty$ and that each $f \in T_X$ satisfies $|f(x) - f(y)| = \big| \|f, h_x\| - \|f, h_y\| \big| \leqslant \|h_x, h_y\| = xy$ for all $x, y \in X$.

**(1.7)** From the continuity condition $|f(x) - f(y)| \leqslant xy$ for all $f \in T_X$ and $x, y \in X$ it follows easily that $T_X$ can be identified with $T_{\bar{X}}$, if $\bar{X}$ denotes the completion of $X$, and that $T_X$—being obviously complete as a metric space—is a compact space if and only if $X$ is precompact, i.e., $\bar{X}$ is compact.

**(1.8)** Next we observe that for $f, g \in T_X$ we have $\sup(f(x) - g(x) \mid x \in X) = \sup(\sup(xy - f(y) - g(x) \mid y \in X) \mid x \in X) = \sup(\sup(xy - f(y) - g(x) \mid x \in X) \mid y \in X) = \sup(g(y) - f(y) \mid y \in X)$ and thus $\|f, g\| = \sup(f(x) - g(x) \mid x \in X) = \sup(g(x) - f(x) \mid x \in X) = \sup(xy - f(y) - g(x) \mid x, y \in X) = \sup(\|h_x, h_y\| - \|f, h_y\| - \|g, h_x\| \mid x, y \in X)$, which in turn implies that $T_X$, considered as an extension of $X$ via the canonical embedding $X \to T_X$: $x \mapsto h_x$, is indeed a tight extension since, more generally, and extension $Y$ of $X$ with $y_1 y_2 = \sup(x_1 x_2 - x_1 y_1 - y_2 x_2 \in X)$ for all $y_1, y_2 \in Y$ is necessarily tight: if $d: Y \times Y \to \mathbb{R}$ satisfies the conditions (D1) and (D3) from the Introduction as well as $d(y_1, y_2) \leqslant y_1 y_2$ for all $y_1, y_2 \in Y$ and $d(x_1, x_2) = x_1 x_2$ for all $x_1, x_2 \in X$, then for all $y_1, y_2 \in Y$ we have $y_1 y_2 = \sup(x_1 x_2 - x_1 y_1 - y_2 x_2 \mid x_1 x_2 \in X) \leqslant \sup(d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2) \mid x_1, x_2 \in X) \leqslant d(y_1, y_2)$ and thus $y_1 y_2 = d(y_1, y_2)$.

Another consequence of our formula $\|f, g\| = \sup(f(x) - g(x) \mid x \in X) = \sup(g(x) - f(x) \mid x \in X)$ is that for $x \in X \neq \{x\}$ and $Y = X \backslash \{x\}$ the restriction map $T_X \to P_Y$: $f \to f^x =: f|_Y$ satisfies $\|f, g\| = \|f^x, g^x\|$ for all $f, g \in T_X$, i.e., it is an isometry and so, in particular, we see once again that it is injective (cf. (1.4)).

**(1.9)** To show that, moreover, $T_X$ is the "universal" tight extension of $X$ and that is contractible as well, we have to show that there exists always a "retraction map" $p: P_X \to T_X$, i.e., a map satisfying the conditions

(a) $\|p(f), p(g)\| \leqslant \|f, g\|$ for all $f, g \in P_X$ and

(b) $p(f) \leqslant f$ for all $f \in P_X$ (and thus $p(f) = f$ for all $f \in T_X$).

The existence of such a map $p$ follows from Zorn's lemma from which we can conclude that the set $\mathscr{P}$ of all maps $p: P_X \to P_X$ satisfying the above conditions (a) and (b) contains minimal elements with respect to the ordering "$p_1 \leqslant p_2 \Leftrightarrow p_1(f) \leqslant p_2(f)$ and $\|p_1(f), p_1(g)\| \leqslant \|p_2(f), p_2(g)\|$ for all $f, g \in P_X$" together with the fact that the various maps $p_x: P_X \to P_X$: $f \mapsto p_x(f)$ $(x \in X)$ (cf. (1.3)) are in $\mathscr{P}$ and thus satisfy $p_x \cdot p = p$ for any minimal $p \in \mathscr{P}$ which in turn implies $p(P_X) \subseteq T_X$ by (1.3) for any such $p \in \mathscr{P}$.

Note that for any such $p: P_X \to T_X$ and any $f \in P_X$ with $f|_Y \in T_Y$ for some $Y \subseteq X$ one has necessarily $f|_Y = p(f)|_Y$ by (1.3), since $p(f)|_Y \in P_Y$ and $p(f)|_Y \leqslant f|_Y \in T_Y$.

In particular for $x \in X$, $Y = X \backslash \{x\}$ and $f \in P_X^x = \{h \in P_X \mid h|_Y \in T_Y\}$ one has $p(f) = p_x(f)$ for any such $p$, since $p(f)|_Y = f|_Y = p_x(f)|_Y$ by the above remark and thus by (1.4): $p(f)(x) = \sup(xy - p(f)(y) \mid y \in Y) = \sup(xy - f(y) \mid y \in Y) = \sup(0, xy - f(y) \mid y \in X) = p_x(f)(x)$ in case $Y \neq \varnothing$, whereas the remark is trivial in case $Y = \varnothing$. Hence, though there are many possible choices for $p$, their images coincide on any $f \in P_X^x$ for any $x \in X$.

*Remark.* It seems worthwile to remark that one can circumvent the use of Zorn's lemma in this context. In fact, one can construct a "canonical" retraction $p: P_X \to T_X$ in the following way:

For each $f$: $X \to \mathbb{R}$ let $f^*$ denote the map $f^*: X \to \mathbb{R} \cup \{\infty\}$: $x \mapsto \sup(xy - f(y) \mid y \in X)$ so that $f \in P_X$ if and only if $f^* \leqslant f$ and $f \in T_X$ if and only if $f^* = f$. Now define $q$: $P_X \to P_X$ by $d(f) = \frac{1}{2}(f + f^*)$. Since $q(f)(x) + q(f)(y) = \frac{1}{2}(f(x) + f^*(x)) + \frac{1}{2}(f(y) + f^*(y)) = \frac{1}{2}(f(x) + f^*(y)) + \frac{1}{2}(f^*(x) + f(y)) \geqslant \frac{1}{2}xy + \frac{1}{2}xy = xy$ for all $x, y \in X$ one has indeed $q(f) \in P_X$ for all $f \in P_X$. Since $f^* \leqslant f$ for $f \in P_X$ one has also $q(f) \leqslant f$ for all $f \in P_X$ and, finally, one has $\|q(f), q(g)\| = \sup(|\frac{1}{2}f(x) + \frac{1}{2}f^*(x) - \frac{1}{2}g(x) - \frac{1}{2}g^*(x)| \mid x \in X) \leqslant \frac{1}{2}\|f, g\| + \frac{1}{2}\|f^*, g^*\| \leqslant \|f, g\|$, since $f^*(x) = \sup(xy - f(y) \mid y \in X) = \sup(xy - g(y) + g(y) - f(y) \mid y \in X) \leqslant g^*(x) + \|g, f\|$ together with $g^*(x) \leqslant f^*(x) + \|g, f\|$ implies $\|f^*, g^*\| \leqslant \|f, g\|$ for all $f, g \in P_X$. Thus $q$ satisfies the conditions (a) and (b) and, so, $q^n$ satisfies the same conditions for all $n \in \mathbb{N}$. It follows that $p = \lim_{n\to\infty} q^n$: $P_X \to P_X$ defined by $p(f)(x) = \lim_{n\to\infty} q^n(f)(x)$ is well defined and satisfies (a) and (b) as well. Moreover, one has $p(f) \in T_X$ for all $f \in P_X$ since $q(f) \in P_X$ and $f^* \leqslant q(f) = \frac{1}{2}(f^* + f) \leqslant f$ together imply $f^* \leqslant q(f)^* \leqslant q(f)$ and thus $q(f)(x) - q(f)^*(x) \leqslant \frac{1}{2}(f(x) - f^*(x))$ which in turn implies $q^n(f)(x) - q^n(f)^*(x) \leqslant (1/2^n)(f(x) - f^*(x))$ and hence $p(f)(x) = p(f)^*(x)$ for all $x \in X$, i.e., one has indeed $p(f) = p(f)^*$ or, equivalently, $p(f) \in T_X$. Note that the same construction can also be used to circumvent the use of Zorn's lemma in (1.3).

**(1.10)** Using $p$ we can define for any $f \in T_X$ a homotopy $[0, 1] \times T_X \to T_X$: $(t, g) \mapsto p_t(g) =: p(t \cdot f + (1-t) \cdot g)$ from $p_0$, the identity on $T_X$, to $p_1$, the constant map $T_X \to \{f\} \subseteq T_X$, hence $T_X$ is contractible. Note that, moreover, for any $s, t \in [0, 1]$ with $s \leqslant t$ one has $\|p_s(g), p_t(g)\| \leqslant \|sf - (1-s)g, tf - (1-t)g\| = (t-s) \cdot \|f, g\|$ which together with $\|f, g\| \leqslant \|f = p_1(g), p_t(g)\| + \|p_t(g), p_s(g)\| + \|p_s(g), g = p_0(g)\|$ implies $\|p_t(g), p_s(g)\| = (t-s) \cdot \|f, g\|$ for all $g \in T_X$ and $0 \leqslant s \leqslant t \leqslant 1$. Thus $[0, \|f, g\|] \to T_X$: $t \mapsto p((t/\|f, g\|)f + ((\|f, g\| - t)/\|f, g\|)g)$ is an isometry of the interval $[0, \|f, g\|]$ into $T_X$, connecting $g$ and $f$.

**(1.11)** Another application of (1.9) is the observation that for any extension $Y$ of $X$ there exists an isometry $\tau$: $T_X \hookrightarrow T_Y$ with $\tau(f)|_X = f$ for all $f \in T_X$: to construct $\tau$ choose some fixed $x \in X$ and some retraction $p$: $P_Y \to T_Y$ satisfying the conditions (a) and (b) in (1.9) and define $\tau$: $T_X \to T_Y$ as the composition of $T_X \to P_Y: f \mapsto f'$ defined by

$$f': Y \to \mathbb{R}: y \mapsto f(y) \qquad \text{if } y \in X,$$
$$\mapsto yx + f(x) \qquad \text{if } y \notin X,$$

and of $p: P_Y \to T_Y$. Since $\tau(f)|_X = p(f')|_X \in P_X$ and $p(f')|_X = f$ we have $\tau(f)|_X = f$ for all $f \in T_X$ according to (1.3) and, since $\|f, g\| = \|\tau(f)|_X, \tau(g)|_X\| \leqslant \|\tau(f), \tau(g)\| = \|p(f'), p(g')\| \leqslant \|f', g'\| = \|f, g\|$ for all $f, g \in T_X$, it follows that $\tau: T_X \hookrightarrow T_Y$ is an isometry.

Note that in the particular case where—interchanging the role of $X$ and $Y$—we have $Y = X \setminus \{x\}$ for some $x \in X$ there is only one map $\tau = \tau_x$: $T_Y \to T_X$ with $\tau(f)|_Y = f$ for all $f \in T_Y$ since the restriction map $T_X \to P_Y$: $f \mapsto f^x =: f|_Y$ is an isometry by (1.8). Hence—in view of the existence of at least one isometry $\tau: T_Y \to T_X$ with $\tau(f)^x = f$ for all $f \in T_Y$—the restriction map defines a bijective isometry between $T_X^x =: \{f \in T_X \mid f^x \in T_Y\} = T_X \cap P_X^x$ and $T_Y$ whose unique inverse is then our map $\tau = \tau_x$: $T_Y \to T_X$, constructed above.

Note also that for $f \in P_X^x$ one has necessarily $p_x(f) = \tau_x(f^x)$ as well as $\tau_x(f^x) = p(f)$ for all retraction maps $p: P_X \to T_X$.

More generally, let $Y \subseteq X$ denote an arbitrary subspace of $X$ and consider for some $g \in T_Y$ the map $g^*: X \to \mathbb{R}: x \mapsto \sup(xy - g(y) \mid y \in Y)$. One has obviously $g^*|_Y = g$ and it follows from the above considerations that $g^*(x) = \inf(f(x) \mid f \in T_X,\ f|_Y = g) = \inf(f(x) \mid f \in P_X,\ f|_Y = g)$. In particular, the following statements are equivalent:

(i) $g^* \in P_X$;

(ii) $g^* \in T_X$;

(iii) for any $f \in T_X$ with $f|_Y = g$ one has $f = g^*$;

(iv) for any $f_1, f_2 \in T_X$ with $f_1|_Y = f_2|_Y = g$ one has $f_1 = f_2$.

Finally assume $Y \subseteq X$, $f \in T_X$, $g \in T_Y$, and $\|f|_Y, g\| \leqslant \varepsilon$ for some $\varepsilon \geqslant 0$. Then there exists some $f' \in T_X$ with $f'|_Y = g$ and $\|f, f'\| \leqslant \varepsilon$; consider at first the map

$$f'': X \to \mathbb{R}: x \mapsto \begin{cases} g(x) & \text{if } x \in Y, \\ f(x) + \varepsilon & \text{if } x \in X \setminus Y. \end{cases}$$

Since $g(x) \geqslant f(x) - \varepsilon$ for $x \in Y$ one has $f'' \in P_X$. Thus we may choose a retraction $p: P_X \to T_X$ according to (1.9) and define $f' =: p(f'')$. Since $f'|_Y \leqslant f''|_Y = g \in T_Y$ and $f'|_Y \in P_Y$ we have indeed $f'|_Y = g$. Moreover, we have $f'(x) \leqslant f''(x) \leqslant f(x) + \varepsilon$ for all $x \in X$ and thus $\|f', f\| \leqslant \varepsilon$ by (1.8).

**(1.12)** Next we observe that for any tight extension $Y \supseteq X$ and any extension $T \supseteq X$ a given contracting map $\psi: Y \to T$ satisfying $\psi(x) = x$ for all $x \in X$ is necessarily an isometry since otherwise the map $d: Y \times Y \to \mathbb{R}$: $(x, y) \mapsto \psi(x)\, \psi(y)$ would contradict the tightness of $Y$.

**(1.13)** Now for a tight extension $Y$ of $X$ consider the restriction map $T_Y \to P_X$: $f \mapsto f|_X$. Choose a retraction $p$: $P_X \to T_X$ satisfying the conditions (a) and (b) in (1.9) and let $\psi$: $T_Y \to T_X : f \mapsto p(f|_X)$ denote the composition of $p$ with this restriction map. According to (1.12), $\psi$ must be an isometry. As above let $\tau$: $T_X \to T_Y$ be an isometric embedding satisfying $\tau(f)|_X = f$ for all $f \in T_X$. Then we have $\psi(\tau(f)) = p(\tau(f)|_X) = p(f) = f$ for all $f \in T_X$ and thus, $\psi$ is necessarily surjective. But a surjective isometry is necessarily an isomorphism. So $\tau$: $T_X \to T_Y$ has to be the inverse isomorphism and thus we have necessarily for any $f \in T_Y$ the formula $f|_X = \tau(\psi(f))|_X = \psi(f) \in T_X$, i.e., the restriction map $T_Y \to P_X : f \mapsto f|_X$ maps $T_Y$ already into as well as onto $T_X$, without having to be composed with the retraction map $p$. Thus, altogether we have proved that for any tight extension $Y$ of $X$ the restriction $T_Y \to T_X$: $f \mapsto f|_X$ induces a canonical isomorphism between $T_Y$ and $T_X$.

**(1.14)** As a first consequence we mention: if $Y$ is a tight extension of $X$, then $h_y|_X$ is in $T_X$ for any $y \in Y$ and the thus well-defined map $Y \to T_X$: $y \mapsto h_y|_X$ is an isometric embedding. Moreover, it is easily seen that it is the only isometric embedding $\psi$: $Y \to T_X$ satisfying $\psi(x) = h_x$ for all $x \in X$ since for any such embedding $\psi$: $Y \to T_X$, any $y \in Y$ and any $x \in X$ one has necessarily $h_y|_X(x) = xy = \|\psi(x), \psi(y)\| = \|h_x, \psi(y)\| = \psi(y)(x)$ and thus one has $\psi(y) = h_y|_X$.

**(1.15)** Another consequence is that $T_X$ is fully spread for any metric space $X$, since $Y = T_X$ is a tight extension of $X$, and, hence, the restriction map $\psi$: $T_Y \simeq T_X$: $f \mapsto f|_X$ is a well-defined isomorphism satisfying $\psi(h_y)(x) = h_y(x) = \|y, h_x\| = y(x)$ for all $y \in Y = T_X$ and $x \in X$, i.e., $\psi$ is the inverse of $Y = T_X \to T_Y$: $y \mapsto h_y$. This in turn implies that $Y = T_X \to T_Y$: $y \mapsto h_y$ is an isomorphism and thus $Y = T_X$ has no proper tight extension, since any such extension embeds isometrically into $T_Y$ by (1.14), i.e., $Y = T_X$ is indeed fully spread.

In particular, a space $X$ is fully spread if and only if the embedding $X \to T_X$: $x \mapsto h_x$ is surjective and thus an isomorphism.

**(1.16)** Some examples: If $\#X = 2$, say $X = \{a, b\}$, then $T_X \simeq [0, ab]$: $f \mapsto f(a)$ is easily seen to be a bijective isometry:

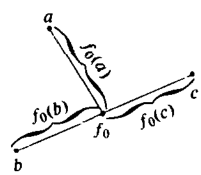
$T_{\{a,b\}} =$ 


In particular, one has $T_X = [h_a, h_b] =: \{(1-t)h_a + th_b \mid t \in [0, 1]\}$.

If $\#X = 3$, say $X = \{a, b, c\}$, then $f_0: X \to \mathbb{R}$, defined by

$$f_0(x) = \frac{ab + ac - cb}{2} \quad \text{if} \quad x = a,$$
$$= \frac{bc + ba - ca}{2} \quad \text{if} \quad x = b,$$
$$= \frac{ca + cb - ab}{2} \quad \text{if} \quad x = c,$$

is the only element $f$ if $T_X$ with $f(x) + f(y) = xy$ for all $x, y \in X$, $T_X \backslash \{f_0\}$ is the disjoint union of the open sets $H_x =: \{f \in T_X \mid f(x) < f_0(x)\}$, $(x \in X)$; $H_x \cup \{f_0\} \cup H_y \to [0, xy] : f \mapsto f(x)$ is a bijective isometry for all $x, y \in X$ with $x \neq y$ and $H_x \cup \{f_0\} = [h_x, f_0]$ for all $x \in X$:

$T_X =$ 


If $\#X = 4$, say $X = \{a, b, c, d\}$, and if, say, $ac + bd \geqslant ab + cd$ and $ac + bd \geqslant ad + bc$, then $T_0 =: \{f \in T_X \mid f(a) + f(c) = ac, \quad f(b) + f(d) = bd\} = \{f \in T_X \mid f(a) + f(b) + f(c) + f(d) = ac + bd\}$ is a closed subset of $T_X$, the map $T_0 \to \mathbb{R} \times \mathbb{R} : f \mapsto ((f(a) + f(b) - ab)/2, (f(a) + f(d) - ad)/2)$ defines a bijective isometry between $T_0$ and the subset $[0, (ac + bd - ab - cd)/2] \times [0, (ac + bd - ad - bc)/2]$ of $\mathbb{R} \times \mathbb{R}$, if $\mathbb{R} \times \mathbb{R}$ is metricized by the "city block metric" $D((s_1, s_2), (t_1, t_2)) = |s_1 - t_1| + |s_2 - t_2|$, $T_X \backslash T_0$ is the disjoint union of the open subsets $H_x =: \{f \in T_X \mid f(x) < k(x)\}$ $(x \in X)$ with

$$k(x) =: \frac{ad + ab - db}{2} \quad \text{if} \quad x = a,$$
$$=: \frac{ba + bc - ac}{2} \quad \text{if} \quad x = b,$$
$$=: \frac{cb + cd - bd}{2} \quad \text{if} \quad x = c,$$
$$=: \frac{dc + da - ca}{2} \quad \text{if} \quad x = d,$$

with

$$f_x : X \to \mathbb{R} : y \mapsto k(x) \quad \text{if} \quad y = x,$$
$$\mapsto xy - k(x) \quad \text{if} \quad y \neq x,$$

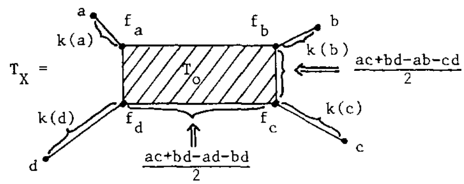



FIGURE A1

one has $f_x \in T_0$ and $H_x \cup \{f_x\} = [h_x, f_x]$. Moreover, $H_x \cup \{f_x\} \to [0, k(x)]$: $f \mapsto f(x)$ is a bijective isometry for each $x \in X$, see Fig. A1. In other words, for any distance map $D: X \times X \to \mathbb{R}$ with $ac + bd \geqslant ab + cd$ and $ac + bd \geqslant ad + bc$, there exist 6 uniquely determined nonnegative numbers $\alpha, \beta, \gamma, \delta, \eta, \zeta$ such that the distance matrix is given by

| $D$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | $\alpha+\eta+\beta$ | $\alpha+\eta+\zeta+\gamma$ | $\alpha+\zeta+\delta$ |
| $b$ | $\alpha+\eta+\beta$ | 0 | $\beta+\zeta+\gamma$ | $\beta+\eta+\zeta+\delta$ |
| $c$ | $\alpha+\eta+\zeta+\gamma$ | $\beta+\zeta+\gamma$ | 0 | $\gamma+\eta+\delta$ |
| $d$ | $\alpha+\zeta+\delta$ | $\beta+\eta+\zeta+\delta$ | $\gamma+\eta+\delta$ | 0 |

in which case $T_X$ is of the form shown in Fig. A2. In particular, $T_X$ is one dimensional if and only if $ab + cd = ac + bd$ or $ad + bc = ac + bd$, and otherwise it is two dimensional. The easy verification of these statements is left to the reader.

If $X = \{a, b, c, d, e\}$ has cardinality 5, there are essentially three "generic" types of metrics, defined on $X$ examples of which are given by the three distance functions

| $D_1$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 9 | 13 | 16 | 10 |
| $b$ | 9 | 0 | 12 | 21 | 17 |
| $c$ | 13 | 12 | 0 | 13 | 17 |
| $d$ | 16 | 21 | 13 | 0 | 10 |
| $e$ | 10 | 17 | 17 | 10 | 0 |

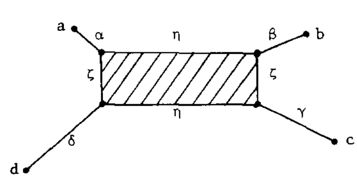



FIGURE A2

| $D_2$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 13 | 21 | 14 | 13 |
| $b$ | 13 | 0 | 10 | 22 | 10 |
| $c$ | 21 | 10 | 0 | 15 | 12 |
| $d$ | 14 | 22 | 15 | 0 | 17 |
| $e$ | 13 | 10 | 12 | 17 | 0 |

| $D_3$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 14 | 16 | 17 | 21 |
| $b$ | 14 | 0 | 28 | 11 | 19 |
| $c$ | 16 | 28 | 0 | 21 | 15 |
| $d$ | 17 | 11 | 21 | 0 | 19 |
| $e$ | 21 | 19 | 15 | 19 | 0 |

The corresponding spaces $T_X$ are shown in Fig. A3. The reader is urged to identify the image(s) of $T_Y$ in $T_X$ for all $Y \subsetneqq X$, in particular for $Y = X \backslash \{x\}$, $(x \in X)$.

In general, in the first case we have 10 nonnegative numbers $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\eta$, $\zeta$, $\vartheta$, $\iota$, $\kappa$ such that the distance matrix is given by

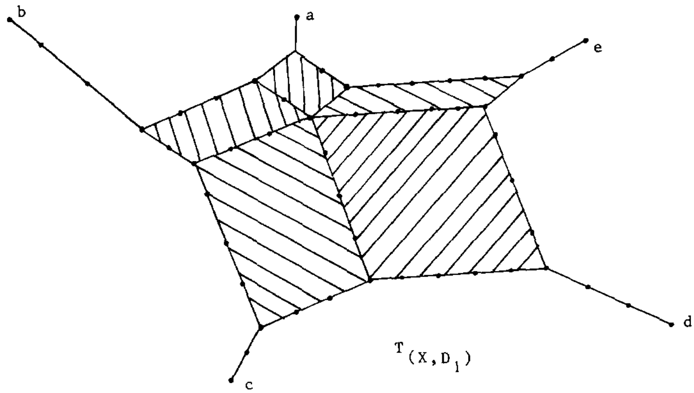
| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | $\alpha+\eta+\zeta+\beta$ | $\alpha+\eta+\zeta+\vartheta+\iota+\gamma$ | $\alpha+\eta+\vartheta+\iota+\kappa+\delta$ | $\alpha+\vartheta+\kappa+\varepsilon$ |
| $b$ | $\alpha+\eta+\zeta+\beta$ | 0 | $\beta+\vartheta+\iota+\gamma$ | $\beta+\zeta+\vartheta+\iota+\kappa+\delta$ | $\beta+\eta+\zeta+\vartheta+\kappa+\varepsilon$ |
| $c$ | $\alpha+\eta+\zeta+\vartheta+\iota+\gamma$ | $\beta+\vartheta+\iota+\gamma$ | 0 | $\gamma+\zeta+\kappa+\delta$ | $\gamma+\eta+\zeta+\iota+\kappa+\varepsilon$ |
| $d$ | $\alpha+\eta+\vartheta+\iota+\kappa+\delta$ | $\beta+\zeta+\vartheta+\iota+\kappa+\delta$ | $\gamma+\zeta+\kappa+\delta$ | 0 | $\delta+\eta+\iota+\varepsilon$ |
| $e$ | $\alpha+\vartheta+\kappa+\varepsilon$ | $\beta+\eta+\zeta+\vartheta+\kappa+\varepsilon$ | $\gamma+\eta+\zeta+\iota+\kappa+\varepsilon$ | $\delta+\eta+\iota+\varepsilon$ | 0 |

and similar descriptions can be given in the other two cases.

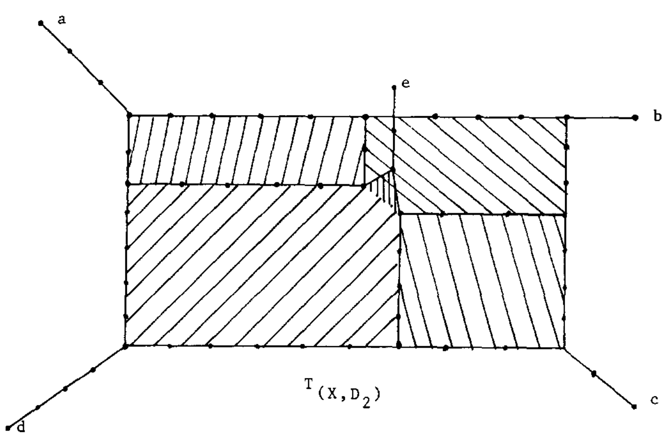
If $X = \{a_1, a_{-1}, a_2, a_{-2}, \dots, a_n, a_{-n}\}$ and

$$a_i a_j = 2 \quad \text{for} \quad i + j \neq 0,$$
$$= 4 \quad \text{for} \quad i + j = 0,$$

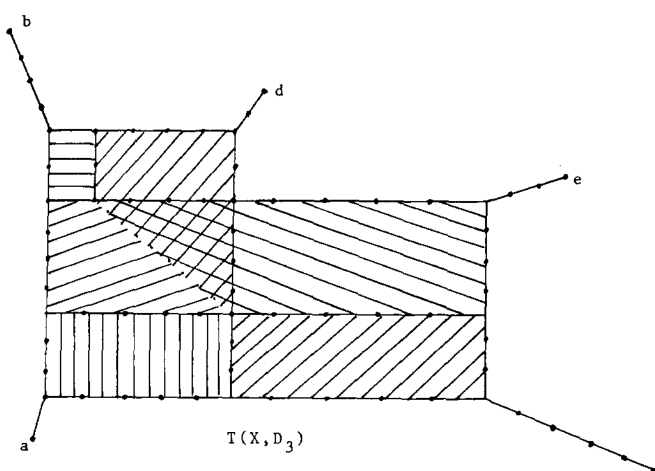
then $\varphi\colon T_X \to \mathbb{R}^n\colon f \mapsto (f(a_1) - 2, f(a_2) - 2, \dots, f(a_n) - 2)$ is an injective isometry if $\mathbb{R}^n$ is metricized by $D((x_1, \dots, x_n), (y_1, \dots, y_n)) = \mathrm{Max}(|x_i - y_i| \mid i = 1, \dots, n)$ and maps $T_X$ onto $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_i| + |x_j| \leqslant 2 \text{ for all } 1 \leqslant i < j \leqslant n\}$, the convex hull of the points $(0, \dots, 0, \pm 2, 0, \dots, 0)$ and $(\pm 1, \pm 1, \dots, \pm 1)$. This follows easily from the fact that $f \in T_X$ implies $f(a_i) + f(a_{-i}) = 4$ for




$T(X, D_1)$




$T(X, D_2)$




$T(X, D_3)$

FIGURE A3

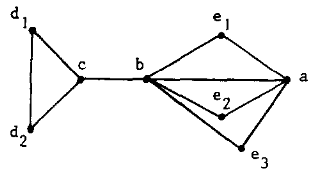



FIGURE A4

all $i = 1,\ldots, n$ since $f(a_i) + f(a_{-i}) \geqslant 4$, $f(a_i) + f(a_j) = 2$, $f(a_{-i}) + f(a_k) = 2$, $f(a_j) \geqslant 0$, and $f(a_k) \geqslant 0$ implies indeed $f(a_i) + f(a_{-i}) = 4$.

As a final example let us consider the space $X = \{a, b, c, d_1, d_2, e_1, e_2, e_3\}$, where the distance $xy$ is given in terms of Fig. A4. If $x$ and $y$ are connected by an edge in this graph, then we put $xy = 2$, otherwise we put $xy = 1$.

Then the space $T_X$ is the union of the 5 subspaces shown in Fig. A5 which are pasted together along the indicated lines. The straight lines $\overline{f_0 f_1}$, $\overline{f_0 f_2}$, $\overline{f_0 f_3}$, $\overline{gd_1}$, and $\overline{gd_2}$ have to be identified. Note that in a natural way $T_X$ has the structure of a 2-dimensional cell complex.

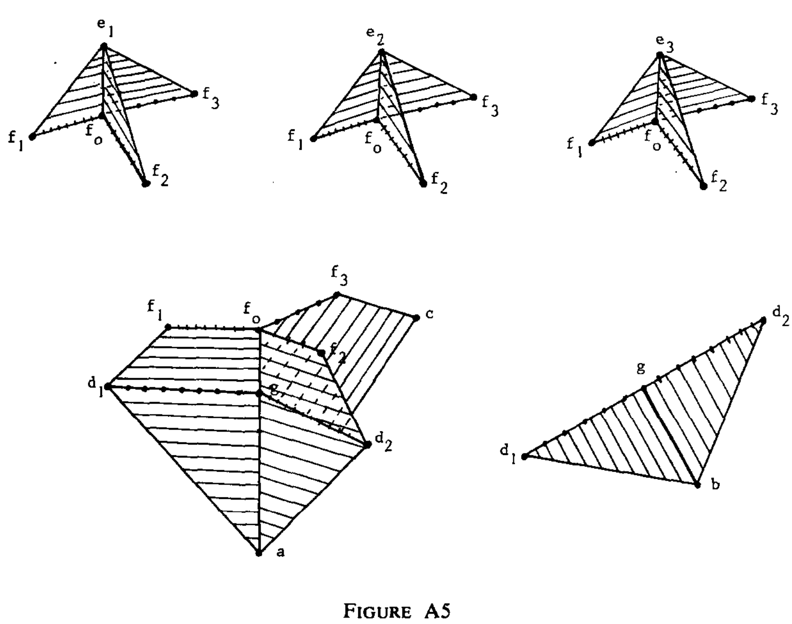



FIGURE A5

## 2. Proofs of Theorems 1–4

We are now ready to prove the above stated Theorems 1–4.

(2.1) *Proof of Theorem* 1. It was shown in (1.8) that an extension $Y$ of $X$ is indeed tight if for all $y_1, y_2 \in Y$ one has

$$y_1 y_2 = \sup(x_1 x_2 - x_1 y_1 - y_2 x_2 \mid x_1, x_2 \in X).$$

Vice versa, if $Y$ is a tight extension of $X$, then it follows from (1.14) that there exists an isometric embedding $\psi: Y \to T_X$ given by $\psi(y)(x) = yx$ for all $y \in Y$ and $x \in X$ and thus one has

$$\begin{aligned} y_1 y_2 &= \| \psi(y_1), \psi(y_2) \| = \sup(x_1 x_2 - \psi(y_1)(x_1) - \psi(y_2)(x_2) \mid x_1, x_2 \in X) \\ &= \sup(x_1 x_2 - y_1 x_1 - y_2 x_2 \mid x_1, x_2 \in X) \qquad \text{for all } y_1, y_2 \in Y. \end{aligned}$$

(2.2) *Proof of Theorem* 2. The implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are trivial. The implication (i) $\Rightarrow$ (iii) is also trivial since $f(y) \geqslant yx$ for all $y \in X$ and some $x \in X$ together with $f(y) = \sup(yz - f(z) \mid z \in X)$ implies $f(y) \leqslant \sup(yz - zx \mid z \in X) \leqslant yx$ and thus $f(y) = yx$ for all $y \in X$. Part (iii) $\Leftrightarrow$ (v) follows from (1.4) and the last remark in (1.5). Part (iv) $\Rightarrow$ (ii) follows also from (1.3) since, by (1.3), there exists for any $f \in P_Y$ some $g \in T_Y$ with $g \leqslant f$ and if $g(y) = yx$ for some $x \in X$ and all $y \in Y$, then one has of course $f(y) \geqslant yx$ for all $y \in Y$ and this $x \in X$. Thus it remains to show that (v) implies (iv). But, using (1.11), there exists for any $f \in T_Y$ some $g = \varphi(f) \in X$ with $g|_Y = f$ and, using the last remark in (1.15), there exists some $x \in T_X$ with $g = h_x$. So we have indeed $f(y) = g(y) = h_x(y) = xy$ for all $y \in Y$ and some $x \in X$.

Now assume $X$ to be compact. If $X$ is fully spread, then the conditions (ii) and (iv) are fulfilled for any subset $Y \subseteq X$, so, in particular, they are fulfilled for any finite subset $Y \subseteq X$. Vice versa, if (ii) or (iv) is fulfilled for any finite subset $Y \subseteq X$ and if for some $f \in P_X$ and any $Y \subseteq X$ we put $Y_f = \{x \in X \mid f(y) \geqslant yx \text{ for all } y \in Y\}$, then $Y_f$ is a closed subset of $X$ and we have $Y_f^1 \cap \cdots \cap Y_f^k = (Y^1 \cup \cdots \cup Y^k)_f \neq \varnothing$ for all finite families $Y^1, \ldots, Y^k \subseteq X$ of finite subsets of $X$ which implies $X_f = \bigcap_{Y \subseteq X, Y \text{ finite}} Y_f \neq \varnothing$, i.e., (i) is fulfilled and, thus, $X$ is fully spread. Finally, it follows from (1.10) that $T_X$ is contractible for any $X$ and, thus, $X$ is contractible whenever $X$ is fully spread. Moreover, also according to (1.10), the explicit homotopy given there satisfies the special conditions stated in Theorem 2.

(2.3) *Proof of Theorem* 3. (i) follows from (1.4), (ii) from (1.5), and (iii) from (1.8) and (1.6); (iv) follows also from (1.6), (v) from (1.3), (vi) from (1.11), and (vii) from (1.13). The following statements in Theorem 3 are by now also obvious or follow from (1.14) and (1.15), except the fact that $T_X$ is compact if and only if $X$ is precompact, i.e., the completion $\bar{X}$ of $X$ is compact. But this follows from (1.7).

(2.4) *Proof of Theorem* 4. Let $X$ be a compact metric space and let $F_X$ denote the set of all $y \in X$ for which there exists some $x \in X$ with $xy + yz > xz$ for all $z \in X \backslash \{y\}$. If $Y \subseteq X$ is a closed subset such that $X$ is a tight extension of $Y$, then $X \to T_Y$: $x \mapsto h_x|_Y$ is an isometry and, thus for $y \in F_X$ and $x \in X$ with $xy + yz > xz$ for all $z \in X \backslash \{y\}$, we have $xy = \| h_x|_Y, h_y|_Y \| = \sup(xz - yz \mid z \in Y) = \max(xz - yz \mid z \in Y) = xz_0 - yz_0$ for some $z_0 \in Y$ which has to coincide with $y$ since otherwise $xy + yz_0 > xz_0$. Thus $y = z_0 \in Y$, i.e., $F_X \subseteq Y$.

Vice versa, if $F_X \subseteq Y \subseteq X$, then $X$ is a tight extension of $Y$ by Theorem 1 since if for $x_1, x_2 \in X$ one chooses $y_1, y_2 \in X$ such that $y_1 y_2 = yx_1 + x_1 x_2 + x_2 y_2$ and such that $y_1 y_2$ is maximal with respect to this property, then one has necessarily $y_1, y_2 \in F_X \subseteq Y$. Otherwise there exists some $z \in X$ with $y_1 y_2 + y_2 z = y_1 z$ and $z \neq y_2$, or with $y_2 y_1 + y_1 z = y_2 z$ and $z \neq y_1$, respectively, which implies $y_1 y_2 < y_1 z = y_1 x_1 + x_1 x_2 + x_2 y_2 + y_2 z \geqslant y_1 x_1 + x_1 x_2 + x_2 z \geqslant y_1 z$ and thus $y_1 y_2 < y_1 z = y_1 x_1 + x_1 x_2 + x_2 z$ or $y_1 y_2 < zy_2 = zx_1 + x_1 x_2 + x_2 y_2$, in contradiction to the maximality of $y_1 y_2$. Thus, $x_1 x_2 = \sup(y_1 y_2 - y_1 x_1 - y_2 x_2 \mid y_1, y_2 \in Y)$ for all $x_1, x_2 \in X$ and, so, $X$ is a tight extension of $Y$ by Theorem 1. The remaining statements of Theorem 4 follow also from this argument.

**(2.5)** Let us consider, finally, another remarkable property of the spaces $T_X$. We define a subset $K \subseteq X$ of a metric space $X$ to be convex, if $x, y \in K$, $z \in X$, and $xy = xz + zy$ implies $z \in K$. Then we have: if $K_1, K_2, \ldots, K_n \subseteq T_X$ are convex subsets of $T_X$ such that $K_i \cap K_j \neq \varnothing$ for all $i, j = 1, \ldots, n$, then $\bigcap_{i=1}^{n} K_i \neq \varnothing$.

*Proof.* Induction with respect to $n$ reduces the proof immediately to the case $n = 3$. Now assume $f_1 \in K_2 \cap K_3$, $f_2 \in K_2 \cap K_3$, and $f_3 \in K_3 \cap K_1$. In $T_{\{f_1, f_2, f_3\}}$ there is some $g$ with $g(f_i) + g(f_j) = \| f_i, f_j \|$ for all $1 \leqslant i < j \leqslant 3$, namely $g(f_i) = \frac{1}{2}(\| f_i, f_j \| + \| f_i, f_k \| - \| f_j, f_k \|)$ whenever $\{1, 2, 3\} = \{i, j, k\}$. Now choose some isometry $T_{\{f_1, f_2, f_3\}} \to T_{(T_X)} = T_X$ which maps $f_i = h_{f_i} \in T_{\{f_1, f_2, f_3\}}$ onto $f_i$ and let $g'$ denote the image of $g$ with respect to this isometry. Then we have $\| g', f_i \| + \| g', f_j \| = g(f_i) + g(f_j) = \| f_i, f_j \|$ for all $1 \leqslant i < j \leqslant 3$ and thus we have $g' \in K_k$ for all $k = 1, 2, 3$, i.e., we have $K_1 \cap K_2 \cap K_3 \neq \varnothing$.

## 3. Optimal Networks

(3.1) *Proof of Theorem* 5. Let $\Gamma = (V, \mathscr{E}, l)$ be an optimal realization of the finite metric space $X \subseteq V$[1]. For any two vertices $u, v \in V$ let $\overline{uv}$ denote the infimum of all sums $l(\{v_0, v_1\}) + \cdots + l(\{v_{n-1}, v_n\})$, where $(v_0, v_1, ..., v_n)$ runs through all finite sequences in $V$ with $v_0 = u$, $v_n = v$ and $\{v_{i-1}, v_i\} \in \mathscr{E}$ for all $i = 1, ..., n$. Then $V \times V \to \mathbb{R} : (u, v) \mapsto \overline{uv}$ defines a metric on $V$ which extends the metric defined on $X$. Now choose some retraction $p: P_X \to T_X$ according to (1.9) and define $\psi: V \to T_X$ by $\psi(v) = p(h_v|_X)$. Then we have $\psi(x) = p(h_x|_X) = h_x$ for all $x \in X$ and $\| \psi(v_1), \psi(v_2) \| \leqslant \overline{v_1 v_2} \leqslant l(\{v_1, v_2\})$ for all $v_1, v_2 \in V$ with $\{v_1, v_2\} \in \mathscr{E}$. Thus we have for all $x, y \in X$ the relation $xy = \| \psi(x), \psi(y) \| \leqslant \inf(\| \psi(v_0), \psi(v_1) \| + \cdots + \| \psi(v_{n-1}), \psi(v_n) \| \mid v_0, v_1, ..., v_n \in V;\ v_0 = x,\ v_n = y;\ \{v_{i-1}, v_i\} \in \mathscr{E}$ for all $i = 1, ..., n) \leqslant \inf(l(\{v_0, v_1\}) + \cdots + l(\{v_{n-1}, v_n\}) \mid v_0, v_1, ..., v_n \in V;\ x = v_0,\ y = v_n;$ $\{v_{i-1}, v_i\} \in \mathscr{E}$ for $i = 1, ..., n) = xy$, so $\Gamma = (V, \mathscr{E}, l')$ with $l' : \mathscr{E} \to \mathbb{R}_+ \cup \{0\}$: $\{u, v\} \mapsto \| \psi(u), \psi(v) \|$ is also a realization of $X$ with $l'(\{u, v\}) \leqslant l(\{u, v\})$ for all $\{u, v\} \in \mathscr{E}$. Thus, the optimality of $\Gamma$ and $l$ implies $l = l'$, i.e., $\| \psi(u), \psi(v) \| = l(\{u, v\})$ for all $u, v \in V$ with $\{u, v\} \in \mathscr{E}$.

(3.2) *Remarks.* (1) Note that for any such $\psi$: $V \to T_X$ with $\| \psi(u), \psi(v) \| = l(\{u, v\})$ for all $u, v \in V$ with $\{u, v\} \in \mathscr{E}$ one necessarily has $\| \psi(u), \psi(v) \| \leqslant \overline{uv}$ for all $u, v \in V$.

(2) It seems reasonable to conjecture that any such $\psi$: $V \to T_X$ necessarily is injective. To support this conjecture let us observe that we have at least $\psi^{-1}(h_x) = \{x\}$ for all $x \in X$: if one chooses for each $y \in X$ a finite sequence $v_0, v_1, ..., v_n \in V$ with $v_0 = y$, $v_n = x$, $\{v_{i-1}, v_i\} \in \mathscr{E}$ $(i = 1, ..., n)$, and $yx = \sum_{i=1}^{n} l(\{v_{i-1}, v_i\})$, and for each pair $(y_1, y_2) \in X^2$ with $y_1 y_2 < y_1 x + x y_2$ a finite sequence $w_0, w_1, ..., w_m \in V$ with $w_0 = y_1$, $w_m = y_2$, $\{w_{i-1}, w_i\} \in \mathscr{E}$ $(i = 1, ..., m)$, and $\sum_{i=1}^{m} l(\{w_{i-1}, w_i\}) = y_1 y_2$, if $V' \subseteq V$ denotes the set of vertices and $\mathscr{E}' \subseteq \mathscr{E}$ the set of edges occurring in these sequences, then it is easy to see that $\Gamma' = (V', \mathscr{E}', l|_{\mathscr{E}'})$ is a realization of $X$, too, and so one has $\Gamma = \Gamma'$ because of the optimality of $\Gamma$. Whereas $V'$ cannot contain any vertex $v \neq x$ with $\psi(v) = h_x$, since for any $v \in V' \backslash \{x\}$, there is either some $y \in X$ with $\| h_y, \psi(v) \| \leqslant \overline{yv} < yx = \| h_y, h_x \|$ or there is a pair $(y_1, y_2) \in X^2$ with $y_1 y_2 \leqslant \| h_{y_1}, \psi(v) \| + \| \psi(v), h_{y_2} \| \leqslant \overline{y_1 v} + \overline{v y_2} \leqslant y_1 y_2 < y_1 x + x y_2 = \| h_{y_1}, h_x \| + \| h_x, h_{y_2} \|$.

(3.3) *Proof of Theorem* 6. Let $X = Y \cup Z$ and $f$: $X \to \mathbb{R}_+$ satisfy the conditions of Theorem 6 (i.e., $Y \times Z \subseteq \mathscr{K}_f$) and put $\mathscr{O}_Y = \{g \in T_X \mid$ there exists some $y \in Y$ with $g(y) < f(y)\}$ and $\mathscr{O}_Z = \{g \in T_X \mid$ there exists some $z \in Z$ with $g(z) < f(z)\}$.

[1] Concerning the existence and finiteness of optimal realizations see the Appendix.

It is clear that $\mathcal{O}_Y$ and $\mathcal{O}_Z$ are open subsets of $T_X$. Since $Y \neq \emptyset \neq Z$ we have $f \in T_X$. Since $g \in T_X$ and $g \neq f$ implies the existence of at least one $x \in X$ with $g(x) < f(x)$ by (1.3) and since $X = Y \cup Z$, we necessarily have $T_X \backslash \{f\} = \mathcal{O}_Y \cup \mathcal{O}_Z$. Finally we have $\mathcal{O}_Y \cap \mathcal{O}_Z = \emptyset$, since $g(y) < f(y)$ and $g(z) < f(z)$ for some $g \in T_X$, $y \in Y$, and $z \in Z$ implies $yz \leqslant g(y) + g(z) < f(y) + f(z) = yz$, a contradiction.

Note that the argument even implies $g(z) > f(z)$ for all $g \in \mathcal{O}_Y$ and $z \in Z$, as well as $g(y) > f(y)$ for all $g \in \mathcal{O}_Z$ and $y \in Y$.

**(3.4)** We now want to indicate how Theorem 5 and Theorem 6 together can be used to derive a result of Imrich and Stotskii (cf. [14]), i.e., we want to prove that for any finite metric space $X$ which admits a nontrivial partition $X = Y \cup Z$ such that there exists some $f \in P_X$ with $f(y) + f(z) = yz$ for all $y \in Y$ and $z \in Z$ and any optimal realization $\Gamma = (V, \mathscr{E}, l)$ of $X \subseteq V$ there is either a vertex $v \in V$ with $\overline{xv} = f(x)$ for all $x \in X$ which occurs in any finite sequence $v_0, v_1, ..., v_n \in V$ with $v_0 \in Y$, $v_n \in Z$, and $\{v_{i-1}, v_i\} \in \mathscr{E}$ $(i = 1, ..., n)$, or there are two vertices $v, w \in V$ with $\overline{yv} < f(y) < yw$ and $\overline{zv} > f(z) > \overline{zw}$ for all $y \in Y$ and $z \in Z$, such that $v, w$ must occur in direct succession in any finite sequence $v_0, v_1, ..., v_n \in V$ with $v_0 \in Y$, $v_n \in Z$, and $\{v_{i-1}, v_i\} \in \mathscr{E}$ $(i = 1, ..., n)$.

So let $\Gamma = (V, \mathscr{E}, l)$ be an optimal realization of $X$ and let $\psi\colon V \to T_X$ be chosen according to Theorem 5. Put $V^Y = \psi^{-1}(\mathcal{O}_Y)$, $V^Z = \psi^{-1}(\mathcal{O}_Z)$, and $V^f = \psi^{-1}(f)$. Since for any $v \in V^Y$ and $w \in V^Z$ with $\{v, w\} \in \mathscr{E}$, one necessarily has $l(\{v, w\}) = \|\psi(v), \psi(w)\| = \|\psi(v), f\| + \|f, \psi(w)\|$, we may introduce for each such pair $(v, w) \in V^Y \times V^Z$ an additional vertex $u_{(v,w)}$, which we use to replace the edge $\{v, w\}$ by the two edges $\{v, u_{(v,w)}\}$ and $\{u_{(v,w)}, w\}$, putting $e(\{v, u_{(v,w)}\}) =: \|\psi(v), f\|$, $l(\{u_{(v,w)}, w\}) =: \|f, \psi(w)\|$, and $\psi(u_{(v,w)}) = f$, this way replacing the original network $\Gamma$ and the original map $\psi$ by another optimal network and another map into $T_X$ which—by abuse of notation—may also be denoted by $\Gamma$ and by $\psi$, and which has the additional property that there is no edge $\{v, w\} \in \mathscr{E}$ with $v \in V^Y$ and $w \in V^Z$.

We claim that for any such optimal network we necessarily have $\# V^f = 1$. This will indeed imply our original claim since it implies that in the original network $\Gamma$ there is either precisely one $u \in V$ with $\psi(u) = f$ and no edge $\{v, w\} \in \mathscr{E}$ with $v \in V^Y$ and $w \in V^Z$, in which case $u$ occurs necessarily in any finite sequence $v_0, v_1, ..., v_n \in V$ with $v_0 \in V^Y$, $v_n \in V^Z$, and $\{v_{i-1}, v_i\} \in \mathscr{E}$ $(i = 1, 2, ..., n)$. In particular, since for any $y \in Y$ and $z \in Z$ there are such sequences with $v_0 = y$, $v_n = z$, and $yz = \sum_{i=1}^n l(\{v_{i-1}, v_i\})$, one necessarily has $yz \leqslant \overline{yu} + \overline{uz} \leqslant \sum_{i=1}^n l(\{v_{i-1}, v_i\}) = yz$, i.e., $yz = \overline{yu} + \overline{uz}$ for all $y \in Y$ and $z \in Z$, and thus $h_u|_X \in T_X$ which implies $h_u|_X = p(h_u|_X) = \psi(u) = f$, i.e., $\overline{ux} = f(x)$ for all $x \in X$. Or there is precisely one edge $\{v, w\} \in \mathscr{E}$ with $v \in V^Y$ and $w \in V^Z$ and no vertex $u \in V$ with $\psi(u) = f$ in which case $v, w$ must occur in direct succession in any finite sequence

$v_0, v_1, \ldots, v_n \in V$ with $v_0 \in Y$, $v_n \in Z$, and $\{v_{i-1}, v_i\} \in \mathscr{E}$ $(i = 1, \ldots, n)$. So, in particular, one has a above, $yz = \overline{yv} + l(\{v, w\}) + \overline{wz} = \overline{yv} + \overline{vz} = \overline{yw} + \overline{wz}$ for all $y \in Y$ and $z \in Z$ which implies $h_v|_X = p(h_v|_X) = \psi(v) \in \mathscr{O}_Y$, and thus $\overline{vy_0} < f(y_0)$ for at least one $y_0 \in Y$ which in turn implies $\overline{vz} = y_0 z - \overline{y_0 v} = f(y_0) + f(z) - \overline{y_0 v} > f(z)$ for all $z \in Z$, and thus $\overline{vy} = yz - \overline{zv} = f(y) + f(z) - \overline{zv} < f(y)$ for all $y \in Y$. The same argument, applied with respect to $w$, yields $\overline{yw} > f(y)$ and $\overline{zw} < f(z)$ for all $y \in Y$ and $z \in Z$.

So let us now assume that $\Gamma = (V, \mathscr{E}, l)$ is an optimal realization of $X \subseteq V$, that $\psi\colon V \to T_X$ is a map which is chosen according to Theorem 5 and that $\Gamma$ and $\psi$ satisfy in addition the condition $\mathscr{E} \cap \{\{v, w\} \mid v \in V^Y = \psi^{-1}(\mathscr{O}_Y), w \in V^Z = \psi^{-1}(\mathscr{O}_Z)\} = \varnothing$. To prove that this implies $\#\psi^{-1}(f) = \#V^f = 1$ let us proceed in several steps:

$\#\psi^{-1}(f) = \#V^f = 1$ let us proceed in several steps:

(a) At first we observe that using (3.2) we may assume $f \notin \psi(X) = \{h_x \mid x \in X\}$.

(b) Second, we associate with $\Gamma$ and $\psi$ the network $\Gamma' = (V', \mathscr{E}', l')$ defined by $V' = \psi(V) = \{\psi(v) \mid v \in V\}$, $\mathscr{E}' = \psi(\mathscr{E}) = \{\{\psi(u), \psi(v)\} \mid \{u, v\} \in \mathscr{E}\}$, and $l'\colon \mathscr{E}' \to \mathbb{R}_+ \colon \{\psi(u), \psi(v)\} \mapsto \|\psi(u), \psi(v)\|$ and observe that—after identification of $X$ and $\psi(X)$, as usual—$\Gamma'$ is a realization of $X$, too, and that one has $\|\Gamma'\| \leqslant \|\Gamma\|$. Thus the optimality of $\Gamma$ implies the optimality of $\Gamma'$ as well as $\|\Gamma'\| = \|\Gamma\|$, i.e., $\{u_1, v_1\}$, $\{u_2, v_2\} \in \mathscr{E}$, and $\{\psi(u_1), \psi(v_1)\} = \{\psi(u_2), \psi(v_2)\}$ implies $\{u_1, v_1\} = \{u_2, v_2\}$.

(c) Next we observe that for any finite sequence $v_0, v_1, \ldots, v_n \in V$ with $v_0 = y \in Y$, $v_n = z \in Z$, and $\{v_{i-1}, v_i\} \in \mathscr{E}$ $(i = 1, \ldots, n)$ there is some $j \in \{0, \ldots, n\}$ with $v_j \in V^f$, so, in particular, we have $V^f \neq \varnothing$.

Moreover, if we have $yz = \sum_{i=1}^n l(\{v_{i-1}, v_i\})$, then we have $yz = f(y) + f(z) = \psi(v_j)(y) + \psi(v_j)(z) \leqslant \overline{yv_j} + \overline{v_j z} \leqslant \sum_{i=1}^n l(\{v_{i-1}, v_i\}) = yz$ and, therefore, we have $\overline{yv_j} = f(y)$ and $\overline{v_j z} = f(z)$.

(d) This implies: if for some $u \in V^f$ and some $y \in Y$ (or $z \in Z$) we have $\overline{yu} = f(y) < \overline{yv}$ (or $\overline{zu} = f(z) < \overline{zv}$, respectively) for all $v \in V^f \backslash \{u\}$, then we have $\overline{uz} = f(z)$ for all $z \in Z$ (or $\overline{uy} = f(y)$ for all $y \in Y$, respectively), since for any sequence $y = v_0, v_1, \ldots, z = v_n \in V$ with $\{v_{i-1}, v_i\} \in \mathscr{E}$ and $\sum_{i=1}^n (\{v_{i-1}, v_i\}) = yz$ the $v_j \in V^f$ with $\overline{yv_j} = f(y)$ and $\overline{zv_j} = f(z)$ necessarily must coincide with $u$.

(e) Next we observe that for any $u \in V^f$ there exists some $w \in V^Y$ (or $w \in V^Z$) with $\{u, w\} \in \mathscr{E}$ once there exists some $y \in Y$ with $\overline{uy} = f(y)$ (or some $z \in Z$ with $\overline{uz} = f(z)$, respectively), since, by (a), we necessarily have $\overline{ux} > 0$ for all $x \in X$.

(f) Now we prove: if $\{u, w\} \in \mathscr{E}$, $u \in V^f$, and $w \in V^Y$ (or $w \in V^Z$), then there exists some $y \in Y$ with $\overline{uy} = f(y) < \overline{vy}$ (or some $z \in Z$ with $\overline{uz} = f(z) < \overline{vz}$, respectively) for all $v \in V^f \setminus \{u\}$.

It is enough to consider the case $w \in V^Y$. Since $\Gamma'$ is optimal, there must exist some $x_1, x_2 \in X$ such that $\psi(w), \psi(u)$ occurs in direct succession in any finite sequence $x_1 = v_0, \ v_1, ..., x_2 = v_n \in V'$ with $\{v_{i-1}, v_i\} \in \mathscr{E}'$ $(i = 1, 2, ..., n)$ and $x_1x_2 = \sum_{i=1}^{n} l'(\{v_{i-1}, v_i\})$, since otherwise we could delete $\{\psi(w), \psi(u)\}$ in $\Gamma'$. This implies in particular $x_1x_2 = \overline{x_1 w} + l(\{w, u\}) + \overline{ux_2} = \overline{x_1 w} + \overline{wx_2} = \overline{x_1 u} + \overline{ux_2} \geqslant \psi(u)(x_1) + \psi(u)(x_2) = f(x_1) + f(x_2) = x_1x_2$ and thus it implies $\psi(w)(x_1) < f(x_1)$ and therefore $x_1 = y \in Y$ since $w \in V^Y$. It implies also $\overline{yu} = \psi(u)(y) = f(y)$ and it implies that there is no $v \in V^f \setminus \{u\}$ with $\overline{yv} = \| \psi(y), \ \psi(v) \| = f(y)$, since otherwise we could find a finite sequence $y = v'_0, \ v'_1, ..., v'_k = v \in V$ with $\{v'_{i-1}, v'_i\} \in \mathscr{E}$ and $\sum_{i=1}^{k} l(\{v'_{i-1}, v'_i\}) = \overline{yv} = f(y)$ and thus in any sequence $x_1 = v_0, v_1, ..., v_{j-1} = \psi(w), \ v_j = \psi(u) = f, \ v_{j+1}, ..., v_n = x_2 \in V'$ with $\{v_{i-1}, v_i\} \in \mathscr{E}'$ $(i = 1, ..., n)$, and $x_1x_2 = \sum_{i=1}^{n} l'(\{v_{i-1}, v_i\})$ we could replace the section $x_1 = v_0, \ v_1, ..., v_j = f$ by $x_1 = y = v'_0 = \psi(v'_0), \ \psi(v'_1), ..., \psi(v'_k) = \psi(v) = f$, this way avoiding the edge $\{\psi(w), \ \psi(u)\}$, in contradiction to our assumption concerning the choice of $x_1$ and $x_2$. Thus we necessarily have $\overline{yv} > \| \psi(y), \psi(v) \| = f(y)$ for all $v \in V^f \setminus \{u\}$.

(g) It is now obvious how to finish the proof: For any $u \in V^f$ there must exist some $w \in V^Y \cup V^Z$ with $\{u, w\} \in \mathscr{E}$. W.l.o.g. we may assume $w \in V^Y$. Then it follows from (f) that there exist some $y = y_0 \in Y$ with $\overline{y_0 u} = f(y_0) < \overline{y_0 v}$ for all $v \in V^g \setminus \{u\}$, which is turn implies, using (d), that we have $\overline{zu} = f(z)$ for all $z \in Z$. Combining this with (e), (f), and (d) we finally get that $\overline{yu} = f(y)$ holds for all $y \in Y$ as well, i.e., one has $\overline{xu} = f(x)$ for all $x \in X$. If there would exist another element $v \in V^f$ with $v \neq u$, we would also have $\overline{vx} = f(x)$ for all $x \in X$, contradicting $\overline{y_0 v} > f(y_0)$. Thus we have indeed $\# V^f = 1$.

*Remark.* The length of this proof makes it even more desirable to prove, in general, that for any optimal realization $\Gamma = (V, \mathscr{E}, l)$ of a finite metric space $X \subseteq V$ the associated maps $\psi\colon V \to T_X$ are necessarily injective, but it shows also that such a proof may be quite complicated.

**(3.5)** Let us now begin with the proof of Theorem 7. We start by analyzing in some detail the network $\Gamma_X = (V_X, \mathscr{E}_X, l_X)$. Recall that for each $f \in P_X$ we have put $\mathscr{K}_f =: \{(x, y) \in X \times X \mid f(x) + f(y) = xy\}$, so that for finite (or compact) $X$ we have $f \in T_X$ if and only if $X = \operatorname{supp} \mathscr{K}_f =: \bigcup_{x, y \in \mathscr{K}_f} \{x, y\}$. For each pair $f, g \in P_X$ let $X_g^f$ denote the set $X_g^f =: \{x \in X \mid f(x) - g(x) = \|f, g\|\}$. The following facts are almost obvious:

(a) For $x, y \in X$ and $f, g \in P_X$ one has $(x, y) \in \mathscr{K}_f$ and $x \in X_g^f$ if and only if one has $(x, y) \in \mathscr{K}_g$ and $y \in X_f^g$, because $f(x) + f(y) = xy$ and

$f(x)-g(x)=\|f,g\|=\sup(|f(z)-g(z)| \mid z\in X)$ together with $g(x)+g(y)\geqslant xy$ implies $\|f,g\| \geqslant g(y)-f(y) \geqslant (xy-g(x))-(xy-f(x)) = f(x)-g(x) = \|f,g\|$ and thus $\|f,g\|=g(y)-f(y)$ as well as $g(x)+g(y)=xy$.

(b) This implies: if $f,g\in P_X$ and if $f\neq g$, then $X_{\{f,g\}}=:X_g^f\cup X_f^g$ satisfies $\mathscr{K}_f\cap\mathscr{K}_g\subseteq X_g^f\times X_f^g\cup X_f^g\times X_g^f\cup(X\backslash X_{\{f,g\}}\times X\backslash_{\{f,g\}})$. In particular, if $f,g\in V_X=:\{h\in T_X \mid X\times X=\bigcup_{n\in\mathbb{N}}\mathscr{K}_h^{2n}\}$ and if $f\neq g$, then $\mathscr{K}_f\neq\mathscr{K}_g$, since $X_g^f\times X_f^g\subseteq X\times X=\bigcup_{n\in\mathbb{N}}\mathscr{K}_f^{2n}$, but $X_g^f\times X_f^g\cap\bigcup_{n\in\mathbb{N}}(\mathscr{K}_f\cap\mathscr{K}_g)^{2n}=\varnothing$. Thus $V_X$ is finite whenever $X$ is finite, since $V_X\to\mathscr{P}(X\times X)\colon f\mapsto\mathscr{K}_f$ is injective, and $\{f,g\}\in\mathscr{E}_X=:\{\{h_1,h_2\}\subseteq V_X \mid h_1\neq h_2$ and $\mathscr{K}_{h_1}\cap\mathscr{K}_{h_2}$ is $X$-connected, i.e., $X\times X=\bigcup_{n\in\mathbb{N}}(\mathscr{K}_{h_1}\cap\mathscr{K}_{h_2})^n\}$ implies $X=X_{\{f,g\}}$, i.e., $\|f,g\|=|f(x)-g(x)|$ for any $x\in X$ as well as

$$\bigcup_{n\in\mathbb{N}}(\mathscr{K}_f\cap\mathscr{K}_g)^{2n+1}=X_g^f\times X_f^g\cup X_f^g\times X_g^f$$

and

$$\bigcup_{n\in\mathbb{N}}(\mathscr{K}_f\cap\mathscr{K}_g)^{2n}=X_g^f\times X_g^f\cup X_f^g\times X_f^g.$$

(c) For $f,g\in P_X$ one has $[f,g]=:\{(1-t)f+tg \mid 0\leqslant t\leqslant 1\}\subseteq P_X$, for each $h\in[f,g]$ one has $\|f,g\|=\|f,h\|+\|h,g\|$ as well as $\mathscr{K}_f\cap\mathscr{K}_g\subseteq\mathscr{K}_h$ and for each $h\in(f,g)=:\{(1-t)f+tg \mid 0<t<1\}$ one has $\mathscr{K}_h=\mathscr{K}_f\cap\mathscr{K}_g$.

**(3.6)** For $\{f,g\}\in\mathscr{E}_X$ one has $[f,g]=\{h\in P_X \mid \|f,h\|+\|h,g\|=\|f,g\|\}=\{h\in P_X \mid \mathscr{K}_f\cap\mathscr{K}_g\subseteq\mathscr{K}_h\}$ and $(f,g)=\{h\in P_X \mid \mathscr{K}_f\cap\mathscr{K}_g=\mathscr{K}_h\}$. In particular, for $\{f_1,g_1\},\{f_2,g_2\}\in\mathscr{E}_X$ one has $\mathscr{K}_{f_1}\cap\mathscr{K}_{g_1}=\mathscr{K}_{f_2}\cap\mathscr{K}_{g_2}$ if and only if $(f_1,g_1)\cap(f_2,g_2)\neq\varnothing$ if and only if $\{f_1,g_1\}=\{f_2,g_2\}$. Moreover, if $\{f,g\}\in\mathscr{E}_X$ and $k_g^f\colon X\to\mathbb{R}$ is defined by

$$\begin{aligned} k_g^f(x) &= -1 \qquad \text{for } x\in X_g^f,\\ &= +1 \qquad \text{otherwise,}\end{aligned}$$

then $g=f+\|f,g\|\cdot k_g^f$ and $[f,g]=\{f+r\cdot k_g^f \mid 0\leqslant t\leqslant\|f,g\|\}$.

*Proof.* From $X_g^f\cup X_f^g=X$ it follows that $g-f=\|f,g\|\cdot k_g^f$ and thus $[f,g]=\{(1-t)f+tg \mid t\in[0,1]\}=\{f+t\cdot\|f,g\|\,k_g^f \mid t\in[0,1]\}=\{f+t\cdot k_g^f \mid 0\leqslant t\leqslant\|f,g\|\}$. The inclusions $[f,g]\subseteq\{h\in P_X \mid \mathscr{K}_f\cap\mathscr{K}_g\subseteq\mathscr{K}_h\}$ and $[f,g]\subseteq\{h\in P_X \mid \|f,h\|+\|h,g\|=\|f,g\|\}$ follow from (3.5(c)). Moreover, $g=f+\|f,g\|\cdot k_g^f$ and $\|f,h\|+\|h,g\|=\|f,g\|$ together imply $\|f,g\|=|f(x)-g(x)|\leqslant|f(x)-h(x)|+|h(x)-g(x)|\leqslant\|f,h\|+\|h,g\|=\|f,g\|$ and thus $|f(x)-h(x)|+|h(x)-g(x)|=|f(x)-g(x)|$ as well as

$|f(x)-h(x)|=\|f,h\|$ and $|h(x)-g(x)|=\|h,g\|$ for all $x\in X$ which in turn easily implies $h=f+\|f,h\|\,k_g^f\in[f,g]$.

So we have indeed $[f,g]=\{f+tk_g^f\,|\,0\leqslant t\leqslant\|f,g\|\}=\{h\in P_X\,\big|\,\|f,h\|+\|h,g\|=\|f,g\|\}$. Now assume $\mathscr{K}_f\cap\mathscr{K}_g\subseteq\mathscr{K}_h$ for some $h\in P_X$. W.l.o.g. assume $f\neq h\neq g$, so $\mathscr{K}_h\cap\mathscr{K}_f\subseteq X_h^f\times X_f^h\cup X_f^h\times X_h^f\cup(X\backslash X_{\{f,h\}}\times X\backslash X_{\{f,h\}})$. But $\mathscr{K}_f\cap\mathscr{K}_g\subseteq\mathscr{K}_f\cap\mathscr{K}_h$ and $\bigcup_{n\in\mathbb{N}}(\mathscr{K}_f\cap\mathscr{K}_g)^{2n}=X_g^f\times X_g^f\cup X_f^g\times X_f^g$. Thus $X_g^f=X_f^h$ or $X_f^g=X_f^h$ and hence $k_h^f=\pm k_g^f$ as well as $h=f+\|f,h\|\,k_h^f$. But in case $k_h^f=-k_g^f$ we get $h\neq g$ and $f=(\|f,g\|/(\|f,g\|+\|f,h\|))\cdot h+(\|f,h\|/(\|f,g\|+\|f,h\|))\cdot g\in(h,g)$, so we have $\mathscr{K}_f=\mathscr{K}_h\cap\mathscr{K}_g\subseteq X_g^h\times X_h^g\cup X_h^g\times X_g^h\cup(X\backslash X_{\{g,h\}}\times X\backslash X_{\{g,h\}})$, in contradiction to $\bigcup_{n\in\mathbb{N}}\mathscr{K}_f^{2n}=X\times X$. Thus we have $k_g^f=k_h^f$ and, similarly, we have $-k_g^f=k_f^g=k_h^g$ which together implies indeed $h\in(f,g)$. In particular, our result and (3.5c) together imply $\mathscr{K}_f\cap\mathscr{K}_g=\mathscr{K}_h$ for each $h\in P_X$ with $\mathscr{K}_f\cap\mathscr{K}_g\subseteq\mathscr{K}_h$ and $f\neq h\neq g$.

**(3.7)** If $X$ is finite (or compact), then for any $h\in P_X$ for which $\mathscr{K}_h$ is $X$-connected and bipartite (i.e., $\mathscr{K}_h$ satisfies $\bigcup_{n\in\mathbb{N}}\mathscr{K}_h^n=X\times X\neq\bigcup_{n\in\mathbb{N}}\mathscr{K}_h^{2n}$) there is precisely one edge $\{f,g\}\in\mathscr{E}_X$ with $h\in(f,g)=[f,g]\backslash\{f,g\}$ and for this edge one has, of course, $\mathscr{K}_h=\mathscr{K}_f\cap\mathscr{K}_g$ as well as

$$[f,g]=\{d\in P_X\,|\,\mathscr{K}_h\subseteq\mathscr{K}_d\},$$
$$\{f,g\}=\{d\in P_X\,|\,\mathscr{K}_h\subsetneqq\mathscr{K}_d\},$$

and

$$(f,g)=\{d\in P_X\,|\,\mathscr{K}_h=\mathscr{K}_d\}.$$

So altogether one has

$$E_X^0=:\{h\in P_X\,|\,\mathscr{K}_h\text{ is connected and bipartite}\}=\dot{\bigcup_{\{f,g\}\in\mathscr{E}_X}}(f,g)$$

and

$$E_X=:\{h\in P_X\,|\,\mathscr{K}_h\text{ is connected}\}=V_X\,\dot{\cup}\dot{\bigcup_{\{f,g\}\in\mathscr{E}_X}}(f,g).$$

*Proof.* Let $X=Y\dot{\cup}Z$ be the unique "bipartition" of $X$ with $\mathscr{K}_h\subseteq Y\times Z\cup Z\times Y$ and let $k\colon X\to\mathbb{R}$ denote the map defined by

$$k(x)=-1\qquad\text{for}\quad x\in Y,$$
$$\quad=+1\qquad\text{for}\quad x\in Z,$$

let

$$a = \min\left(\frac{h(y_1)+h(y_2)-y_1y_2}{2} \;\middle|\; y_1, y_2 \in Y\right),$$

$$b = \min\left(\frac{h(z_1)+h(z_2)-z_1z_2}{2} \;\middle|\; z_1, z_2 \in Z\right),$$

and put $f = h + ak$, $g = h - bk$. Then one easily verifies $f, g \in P_X$, $\mathscr{K}_h \subseteq \mathscr{K}_f \cap \mathscr{K}_g$, $\mathscr{K}_f \cap (Y \times Y) \neq \varnothing$, and thus $f \in V_X$, $\mathscr{K}_g \cap (Z \times Z) \neq \varnothing$, and thus $g \in V_X$, $\{f, g\} \in \mathscr{E}_X$, $k = k_g^f$, and $h \in (f, g)$. Thus our claim follows easily from (3.6).

**(3.8)** Now we want to study for any $f \in V_X$ the set $N_f = \{g \in V_X \mid \{f, g\} \in \mathscr{E}_X\}$. For this purpose we consider symmetric relations $\mathscr{K} \subseteq X \times X$ and subsets $Y \subseteq X$ and define $Y$ to be $\mathscr{K}$-admissible, if $(Y \times Y) \cap \mathscr{K} = \varnothing$ and $\mathscr{K}_Y =: \mathscr{K} \cap (X \times Y \cup Y \times X)$ is $X$-connected, so $\mathscr{K}_Y$ is $X$-connected and bipartite and $X = Y \mathbin{\dot\cup} (X \backslash Y)$ is the unique bipartition of $X$ with respect to $\mathscr{K}_Y$.

Now we claim: if $X$ is finite and if $\{f, g\} \in \mathscr{E}_X$ then $X_g^f$ is $\mathscr{K}_f$-admissible and the map

$$N_f \to \mathscr{P}(X) : g \mapsto X_g^f$$

defines a bijection between $N_f$ and the set $\mathscr{Y}_f$ of $\mathscr{K}_f$-admissible subsets of $X$. Moreover, if $g \in N_f$ corresponds to $Y = X_g^f$, then $\mathscr{K}_f \cap \mathscr{K}_g$ coincides with $(\mathscr{K}_f)_Y$.

*Proof.* Since $\{f, g\} \in \mathscr{E}_X$ implies that $X = X_g^f \mathbin{\dot\cup} X_f^g$ is the unique bipartition of $X$ with respect to $\mathscr{K}_f \cap \mathscr{K}_g$ we necessarily have

$$\mathscr{K}_f \cap \mathscr{K}_g \subseteq (\mathscr{K}_f)_{X_g^f} \quad \text{so} \quad (\mathscr{K}_f)_{X_g^f} \text{ is } X\text{-connected.}$$

Moreover, we necessarily have $(X_g^f \times X_g^f) \cap \mathscr{K}_f = \varnothing$, since by (3.5a) we know that $(x, y) \in \mathscr{K}_f$ and $x \in X_g^f$ implies $y \in X_f^g$, and thus $y \notin X_g^f$. Hence $X_g^f$ is $\mathscr{K}_f$-admissible. It follows also from (3.5a) that $(x, y) \in \mathscr{K}_f$ and $x \in X_g^f$ implies $(x, y) \in \mathscr{K}_g$, thus $\mathscr{K}_f \cap \mathscr{K}_g = (\mathscr{K}_f)_{X_g^f}$.

So it remains to show that for any $\mathscr{K}_f$-admissible $Y \subseteq X$ there is a unique $g \in N_f$ with $X_g^f = Y$.

Let $k: X \to \mathscr{R}$ denote the map defined by

$$k(x) = -1 \quad \text{if } x \in Y,$$
$$= +1 \quad \text{otherwise,}$$

let

$$a = \min\left(\frac{f(y_1)+f(y_1)-y_1y_2}{2} \;\middle|\; y_1, y_2 \in Y\right),$$

so $a > 0$ since $X$ is finite and $(Y \times Y) \cap \mathscr{K}_f = \varnothing$, and let $g = f + ak$. Then we have $g \in P_X$, $(\mathscr{K}_f)_Y \subseteq \mathscr{K}_g$, and $\mathscr{K}_g \cap (Y \times Y) \neq \varnothing$, so $\mathscr{K}_g \cap \mathscr{K}_f \supseteq (\mathscr{K}_f)_Y$ is $X$-connected and $\mathscr{K}_g$ is $X$-connected and nonbipartite, i.e., one has $g \in V_X$ and $\{f, g\} \in \mathscr{E}_X$ and one obviously has $X^f_g = \{x \in X \mid k(x) = -1\} = Y$. Finally, if $\{f, g\}$, $\{f, h\} \in \mathscr{E}_X$ and $X^f_g = X^f_h = Y$, then one necessarily has $k = k^f_g = k^h_g$ and thus $g = f + \|f, g\| \cdot k$ and $h = f + \|f, h\| \cdot k$. So one has either $g = h$ or $g \in (f, h)$ or $h \in (f, g)$. But the last two possibilities are ruled out by (3.5b), and (3.5c) since they would imply that either $\mathscr{K}_g$ or $\mathscr{K}_h$ is bipartite in contradiction to $g, h \in V_X$. Thus $g$, $h \in N_f$ and $X^f_g = X^f_h = Y$ implies indeed $g = h$.

**(3.9)** Concerning the existence of $\mathscr{K}_f$-admissible subsets $Y \subseteq X$ we claim:

For any $X$-connected symmetric relation $\mathscr{K} \subseteq X \times X$ with "$(x, x) \in \mathscr{K} \Leftrightarrow (x, z) \in \mathscr{K}$ for all $z \in X$" and any pair of subsets $Y_0 \subseteq X_0 \subseteq X$ such that $\mathscr{K}_0 =: (X_0 \times X_0) \cap \mathscr{K}$ is $X_0$-connected and $Y_0$ is $\mathscr{K}_0$-admissible and nonempty there exists some $\mathscr{K}$-admissible subset $Y \subseteq X$ with $Y \cap X_0 = Y_0$.

*Proof.* Let $X_1 \subseteq X$ be a maximal subset of $X$ containing $X_0$ such that $\mathscr{K}_1 =: (X_1 \times X_1) \cap \mathscr{K}$ is $X_1$-connected and there exists some $\mathscr{K}_1$-admissible subset $Y_1 \subseteq X_1$ with $X_0 \cap Y_1 = Y_0$. We have to show that $X_1 = X$. Otherwise there is some $x \in X \backslash X_1$ and with $Z =: \{y \in X_1 \mid (x, y) \in \mathscr{K}\} \neq \varnothing$. Put $X_2 = X_1 \cup \{x\}$ and put

$$\begin{aligned} Y_2 &= Y_1 && \text{if} \quad Z \cap Y_1 \neq \varnothing, \\ &= Y_1 \cup \{x\} && \text{if} \quad Z \cap Y_1 = \varnothing. \end{aligned}$$

Then it is easy to see that $\mathscr{K}_2 =: (X_2 \times X_2) \cap \mathscr{K}$ is $X_2$-connected and $Y_2$ is an $\mathscr{K}_2$-admissible subset of $X_2$ (here we need our special assumption "$(x, x) \in \mathscr{K} \Leftrightarrow (x, z) \in \mathscr{K}$ for all $z \in X$" which implies $(x, x) \in \mathscr{K}$ in case $Z \cap Y_1 = \varnothing$) with $Y_2 \cap X_0 = Y_0$, contradicting the maximality of $X_1$. Hence our claim is proved.

**(3.10)** It follows easily that for any $f \in V_X$ and any $x \in X$ with $f(x) \neq 0$ there is some $g \in V_X$ with $\{f, g\} \in \mathscr{E}_X$ and $\|f, g\| + g(x) = f(x)$, i.e., with $x \in X^f_g$, since $(z, z) \in \mathscr{K}_f$ for some $z \in X$ implies $f = h_z$ by (1.4), and thus $(z, w) \in \mathscr{K}_f$ for all $w \in X$. So starting with $X_0 = Y_0 = \{x\}$ we can find some $\mathscr{K}_f$-admissible $Y \subseteq X$ with $x \in Y$ which implies $x \in X^f_g$ if $g \in N_f$ is chosen according to (3.8) such that $X^f_g = Y$. It follows that the distance $\overline{fh_x}$ of $f$ and $h_x$ in $\Gamma_x$, defined by $\overline{fh_x} =: \inf(\sum_{i=1}^n l_X(\{v_{i-1}, v_i\}) \mid n \in \mathbb{N}; f = v_0, v_1, \ldots, h_x = v_n \in V_X;\ \{v_0, v_1\}, \ldots, \{v_{n-1}, v_n\} \in \mathscr{E}_X)$ is smaller than or equal to $f(x) = \|f, h_x\|$, and since it cannot be smaller by the triangular inequality one necessarily has $f(x) = \|f, h_x\| = \overline{fh_x}$.

In particular, it follows that for a finite space $X$ the network $\Gamma_X = (V_X, \mathscr{E}_X, l_X)$ satisfies $V_X = \operatorname{supp} \mathscr{E}_X$ and is indeed a realization of $X$ (identified with $\{h_x \mid x \in X\}$ as usual), since $\overline{h_x h_y} = h_x(y) = xy$ for all $x, y \in X$.

From $V_X = \operatorname{supp} \mathscr{E}_X$ we also get that $E_X = V_X \,\dot\cup\, \dot\bigcup_{\{f,g\} \in \mathscr{E}_X} (f, g)$ coincides with $\bigcup_{\{f,g\} \in \mathscr{E}_X} [f, g]$ and thus is coincides with the closure $\overline{E_X^0}$ of $E_X^0$.

It also follows from (3.9) that $\# N_f \geqslant 3$ for all $f \in V_X \backslash X$, since for any such $f \in V_X \backslash X$ one has $(x, x) \notin \mathscr{K}_f$ for all $x \in X$, so $\mathscr{K}_f$—being nonbipartite—contains a smallest odd cycle of length $\geqslant 3$, i.e., there is a subset $X_0 = \{x_1, ..., x_{2k+1}\}$ with $k \geqslant 1$ and $\mathscr{K}_0 = (X_0 \times X_0) \cap \mathscr{K}_f = \{(x_1, x_2), (x_2, x_1), (x_2, x_3), (x_3, x_2), ..., (x_{2k+1}, x_1), (x_1, x_{2k+1})\}$, so that $\mathscr{K}_0$ is $X_0$-connected and any subset $Y_0^i = \{x_i, x_{i+2}, ..., x_{i+2k-2} = x_{i-3}\}$ (with the indices taken modulo $2k+1$) is $\mathscr{K}_0$-admissible for $i = 1, 2, ..., 2k+1$. Thus we have at least $2k + 1 \geqslant 3$ different $\mathscr{K}_f$-admissible subsets $Y \subseteq X$.

**(3.11)** Now let $X$ be finite and let $U \subset E_X^0$ be a finite subset of $E_X^0 = \{h \in P_X \mid \mathscr{K}_h$ is connected and bipartite$\}$. By (3.7) one has $U = \dot\bigcup_{\{f,g\} \in \mathscr{E}_X} (U \cap (f, g)) = \dot\bigcup_{\{f,g\} \in \mathscr{E}_X} \{h \in U \mid \mathscr{K}_h = \mathscr{K}_f \cap \mathscr{K}_g\}$. In particular, for any $\{f, g\} \in \mathscr{E}_X$ there are finitely many real numbers $t_1, ..., t_n$ with $t_0 =: 0 < t_1 < \cdots < t_n < t_{n+1} =: \|f, g\|$ such that $U \cap [f, g] = U \cap (f, g) = \{h \in U \mid \mathscr{K}_h = \mathscr{K}_f \cap \mathscr{K}_g\} = \{f + t_i k_g^f \mid i = 1, ..., n\}$. Note that one has $(h_1, h_2) \cap U = \varnothing$ or, equivalently, $(h_1, h_2) \cap (U \cup V_X) = \varnothing$ for some $h_1, h_2 \in U_{[f,g]} =: \{f, g\} \cup (U \cap (f, g)) = [f, g] \cap (U \cup V_X)$ with $h_1 \neq h_2$ if and only if $\{h_1, h_2\} \in \{\{f + t_{i-1} k_g^f, f + t_i k_g^f\} \mid i = 1, ..., n+1\}$. Thus $\Gamma_{[f,g]}^U =: (U_{[f,g]}, \mathscr{E}_{[f,g]}^U, l_{[f,g]}^U)$ defined by $\mathscr{E}_{[f,g]}^U = \{\{h_1, h_2\} \subseteq U_{[f,g]} \mid h_1 \neq h_2$ and $(h_1, h_2) \cap U = \varnothing\} = \{\{h_1, h_2\} \subseteq U_{[f,g]} \mid h_1 \neq h_2$ and $(h_1, h_2) \cap (U \cup V_X) = \varnothing\}$ and $l_{[f,g]}^U : \mathscr{E}_{[f,g]}^U \to \mathbb{R}_+ : \{h_1, h_2\} \mapsto \|h_1, h_2\|$ is an optimal realization of the space $\{f, g\} \subseteq T_X$, in particular, one has $\|\Gamma_{[f,g]}^U\| = \|f, g\|$. Now we associate with each finite $U \subseteq E_X^0$ the network $\Gamma_X^U = (V_X^U =: V_X \,\dot\cup\, U, \mathscr{E}_X^U, l_X^U)$ defined by $\mathscr{E}_X^U =: \{\{h_1, h_2\} \subseteq V_X^U \mid h_1 \neq h_2,\ \mathscr{K}_{h_1} \cap \mathscr{K}_{h_2}$ is connected and $(h_1, h_2) \cap V_X^U = \varnothing\}$, and $l_X^U : \mathscr{E}_X^U \text{—} \mathbb{R}_+ : \{h_1, h_2\} \mapsto \|h_1, h_2\|$. It follows immediately from the above considerations that $\|\Gamma_X\| = \|\Gamma_X^U\|$, that $\Gamma_X^U$ realizes $X$ the same way as $\Gamma_X$ realized $X$, i.e., that the distance $\overline{h_1 h_2}$—as measured in $\Gamma_X^U$ for $h_1, h_2 \in V_X^U$—satisfies $\overline{h h_x} = h(x)$ for all $h \in V_X^U$ and that the standard construction of eliminating step by step vertices $v \in V_X^U \backslash X$ of degree 2 in $\Gamma_X^U$—which are just the vertices $v \in U$—and replacing the corresponding edges $\{v, u\}$, $\{v, w\}$ by the edge $\{u, w\}$ of length $l(\{u, w\}) =: l(\{v, u\}) + l(\{v, w\})$ leads back to $\Gamma_X$ from $\Gamma_X^U$ for each $U \subseteq E_X^0$: actually, eliminating just one $u \in U$ and replacing the corresponding edges leads to $\Gamma_X^{U \backslash \{u\}}$. Note that in generalization of (3.7) one has

$$E_X^0 \cup V_X = E_X = \bigcup_{\{f,g\} \in \mathscr{E}_X^U} [f, g],$$

$$V_X^U = \operatorname{supp} \mathscr{E}_X^U$$

and

$$E_X \backslash V_X^U = E_X^0 \backslash U = \dot{\bigcup}_{\{f,g\} \in \mathscr{E}_X^U} (f, g),$$

and for $U \subset W \subset E_X^0$, $\# W < \infty$,

$$e = \{f, g\} \in \mathscr{E}_X^U, \qquad \text{and} \qquad \mathscr{E}_e^W =: \{\{u, v\} \in \mathscr{E}_X^W \mid u, v \in [f, g]\},$$

one has

$$\bigcup_{\{u,v\} \in \mathscr{E}_e^W} [u, v] = [f, g],$$

$$\dot{\bigcup}_{\{u,v\} \in \mathscr{E}_e^W} (u, v) = [f, g] \backslash W,$$

and

$$\sum_{\{u,v\} \in \mathscr{E}_e^W} l_X^W(\{u, v\}) = \sum_{\{u,v\} \in \mathscr{E}_e^W} \|u, v\| = \|f, g\| = l_X^U(\{f, g\}).$$

**(3.12)** It is now possible to state Theorem 7 in a more precise form which is well adapted for a proof by induction with respect to $\# X$:

THEOREM 7′. *If $X$ is a finite metric space and $U \subseteq E_X^0$ is a finite subset, then $\Gamma_X^U$ is a hereditarily optimal realization of $X$ and any hereditarily optimal realization $\Gamma = (V, \mathscr{E}, l)$ of $X$ is canonically isomorphic to one such $\Gamma_X^U$. More precisely, if $\Gamma = (V, \mathscr{E}, l)$ is a hereditarily optimal realization of $X \subseteq V$, then $V$ is finite and, if $V \times V \to \mathbb{R}$: $(u, v) \mapsto \overline{uv} =: \min(\sum_{i=1}^n l(\{v_{i-1}, v_i\}) \mid n \in \mathbb{N};\ v_0 = u, v_1, ..., v = v_n \in V;\ \{v_0, v_1\}, \{v_1, v_2\}, ..., \{v_{n-1}, v_n\} \in \mathscr{E})$ is defined as usual, then the map $\varphi_\Gamma: V \to P_X: v \mapsto \varphi_\Gamma(v)$ defined by $\varphi_\Gamma(v)(x) = \overline{vx}$ maps $V$ injectively into $E_X = V_X \dot{\cup} E_X^0$, contains $V_X$ in its image, and defines an isomorphism between $\Gamma$ and $\Gamma_X^U$ with $U = \varphi_\Gamma(V) \backslash V_X$. That is, for the bijection $\varphi = \varphi_\Gamma: V \simeq V_X^U$ one has $\{u, v\} \in \mathscr{E}$ if and only if $\{\varphi(u), \varphi(v)\} \in \mathscr{E}_X^U$ in which case one has $l(\{u, v\}) = l_X^U(\{\varphi(u), \varphi(v)\}) = \|\varphi(u), \varphi(v)\|$.*

**(3.13)** As a first step in the proof of Theorem 7′ let us remark that if $\Gamma = (V, \mathscr{E}, l)$ is a realization of $X \subseteq V$, if $U \subset E_X^0$ is a finite subset, and if $\varphi$: $V \simeq V_X \dot{\cup} U = V_X^U$ is a bijection with $\varphi(x) = h_x$ which induces an isomorphism between $\Gamma$ and $\Gamma_X^U$, then we necessarily have $\varphi = \varphi_\Gamma$, i.e., $\varphi(v)(x) = \overline{vx} = \varphi_\Gamma(v)(x)$ for all $v \in V$ and $x \in X$, since we have $\overline{vx} = \overline{\varphi(v)\, \varphi(x)} = \overline{\varphi(v)\, h_x} = \|\varphi(v), h_x\| = \varphi(v)(x)$. Thus for any realization $\Gamma = (V, \mathscr{E}, l)$ of $X \subseteq V$ there is at most one finite subset $U \subseteq E_X^0$ with $\Gamma \cong_X \Gamma_U^X$, where $\cong_X$ means the existence of an isomorphism $\Gamma \simeq \Gamma_U^X$ given by some $\varphi$:

$V \simeq V_X^U$ which respect $X$, i.e., which satisfies $\varphi(x) = h_x$. If such an $U \subseteq E_X^0$ exists, then there exists precisely one isomorphism $\varphi: \Gamma \simeq \Gamma_U^X$ which respects $X$, namely $\varphi = \varphi_\Gamma$.

**(3.14)** Next let us study the relation between $\Gamma_X$ and $\Gamma_Y$ with $Y = X\backslash\{x\}$ for some $x \in X$. From (1.8) and (1.11) we know that the restriction map $T_X \to P_Y: f \mapsto f^x =: f|_Y$ is an isometry and defines a bijective isometry between $T_X^x = \{f \in T_X \mid f^x \in T_Y\} = T_X \cap P_X^x$ and $T_Y$, whose inverse is denoted by $\tau = \tau_x$: $T_Y \simeq T_X^x \subseteq T_X$ and satisfies $\tau(f^x) = p_x(f) = p(f)$ for any $f \in P_X^x = \{f \in P_X \mid f^x \in T_Y\}$ and any retraction $p: P_X \to T_X$.

Note that $\tau(V_Y) \subseteq V_X$ and $\tau(E_Y) \subseteq E_X$, since a symmetric relation $\mathscr{K} \subseteq X \times X$ with $\mathscr{K} \nsubseteq Y \times Y \cup \{(x, x)\}$ for which $\mathscr{K}^x =: (Y \times Y) \cap \mathscr{K}$ is $Y$-connected (and nonbipartite) is necessarily $X$-connected (and nonbipartite). Put $E^x =: \tau(E_Y) = \{f \in T_X \mid f^x \in E_Y\} = \{f \in T_X \mid \mathscr{K}_f^x = \mathscr{K}_{f^x} \text{ is } Y\text{-connected}\}$,

$$V^x =: V_X \cap E^x = \{f \in V_X \mid f^x \in E_Y\},$$

$$V_Y^x =: \tau^{-1}(V^x) = \{f^x \mid f \in V^x\} = \{f^x \mid f \in V_X\} \cap E_Y,$$

$$U^x =: V_Y^x \backslash V_Y = V_Y^x \cap E_Y^0,$$

$$\Gamma_Y^x =: \Gamma_Y^{U^x},$$

$$\mathscr{E}^x =: \{\{f, g\} \in \mathscr{E}_X \mid \mathscr{K}_f^x \cap \mathscr{K}_g^x \quad \text{is } Y\text{-connected}\}$$
$$= \{\{f, g\} \in \mathscr{E}_X \mid [f^x, g^x] \subseteq E_Y\},$$

$$l^x =: l_X|_{\mathscr{E}^x},$$

and

$$\Gamma^x =: (V^x, \mathscr{E}^x, l^x).$$

So we have, for example,

$$E^x \subseteq E_X, \qquad \operatorname{supp} \mathscr{E}^x \subseteq V^x, \qquad V_Y \subseteq V_Y^x = V_Y \,\dot\cup\, U^x = V_Y^{U^x},$$

and

$$\{h_y \mid y \in Y\} \subseteq \tau(V_Y) \subseteq V^x = \tau(V_Y^x) = \tau(V_Y^{U^x}) = \tau(V_Y) \,\dot\cup\, \tau(U^x) \subseteq V_X.$$

We claim that $\Gamma^x$ is a realization of $Y$ (identified with $\{h_y \mid y \in Y\}$ as usual) and that the restriction map $\varphi$: $V^x \simeq V_Y^x: f \mapsto f^x$ defines an isomorphism $\Gamma^x \simeq \Gamma_Y^x$ which of course satisfies $\varphi(y) = \varphi(h_y) = h_y^x = h_y \in T_Y$ for each $y \in Y$ and thus, using (3.13), it coincides with the map $\varphi_{\Gamma^x}$: $V^x \to P_Y$ defined by $\varphi_{\Gamma^x}(f)(y) = \overline{fh_y}$.

Since $\varphi$: $V^x \simeq V_Y^x: f \mapsto f^x$ is injective by (1.4) or (1.8) and hence a bijection by definition, and since it satisfies $\|f, g\| = \|\varphi(f), \varphi(g)\|$ for all

$f, g \in V^x$ by (1.8), it is enough to show that $\{f, g\} \in \mathscr{E}^x$ if and only if $\{\varphi(f), \varphi(g)\} = \{f^x, g^x\} \in \mathscr{E}^x_Y$.

To this end we use (3.7) and compare the partition

$$E_Y = V_Y \,\dot{\cup} \bigcup^{\bullet}_{\{u,v\} \in \mathscr{E}_Y} (u, v),$$

with the partition

$$E^x = E^x \cap E_X = E^x \cap \left( V_X \,\dot{\cup} \bigcup^{\bullet}_{\{f,g\} \in \mathscr{E}_X} (f, g) \right) = V^x \,\dot{\cup} \bigcup^{\bullet}_{\{f,g\} \in \mathscr{E}^x} (f, g),$$

where the last equality follows from

$$\begin{aligned} E^x \cap (f, g) &= \{h \in E_X \mid \mathscr{K}_h = \mathscr{K}_f \cap \mathscr{K}_g \text{ and } \mathscr{K}^x_h \text{ } Y\text{-connected}\} \\ &= (f, g) \qquad \text{if} \quad \{f, g\} \in \mathscr{E}^x, \\ &= \varnothing \qquad\quad \text{if} \quad \{f, g\} \notin \mathscr{E}^x, \end{aligned}$$

for each $\{f, g\} \in \mathscr{E}_X$, which partition of $E^x$—applying the restriction map—yields

$$E_Y = V^x_Y \,\dot{\cup} \bigcup^{\bullet}_{\{f,g\} \in \mathscr{E}^x} (f^x, g^x).$$

Since $V_Y \subseteq V^x_Y$ it follows that the connected set $(f^x, g^x)$ is a subset of the disjoint union of open sets $E^0_Y = \dot{\bigcup}_{\{u,v\} \in \mathscr{E}_Y} (u, v)$ for each $\{f, g\} \in \mathscr{E}^x$, so for each such $\{f, g\} \in \mathscr{E}^x$ there is precisely one $\{u, v\} \in \mathscr{E}_Y$ which $(f^x, g^x) \cap (u, v) \neq \varnothing$ and in this case one has $(f^x, g^x) \subseteq (u, v)$.

Moreover it follows that for each $\{u, v\} \in \mathscr{E}_Y$ one has $(u, v) = ((u, v) \cap U^x) \,\dot{\cup}\, \dot{\bigcup}_{\{f,g\} \in \mathscr{E}^x, (f^x, g^x) \cap (u,v) \neq \varnothing} (f^x, g^x)$.

It is now easy to show that for $f, g \in V^x$ we have $\{f, g\} \in \mathscr{E}^x$, i.e., $\{f, g\} \in \mathscr{E}_X$ and $\mathscr{K}^x_f \cap \mathscr{K}^x_g$ $Y$-connected, if and only if $\{f^x, g^x\} \in \mathscr{E}^x_Y$, i.e., $f^x \neq g^x$, $\mathscr{K}^x_f \cap \mathscr{K}^x_g$ $Y$-connected and $(f^x, g^x) \cap V^x_Y = \varnothing$: for $\{f, g\} \in \mathscr{E}^x$ we surely have $f \neq g$. Hence, $f^x \neq g^x$ as well as $(f^x, g^x) \cap V^x_Y = \varnothing$, in view of the above partition $E_Y = V^x_Y \,\dot{\cup}\, \dot{\bigcup}_{\{f,g\} \in \mathscr{E}^x} (f^x, g^x)$. Vice versa, $f^x \neq g^x$, the $Y$-connectedness of $\mathscr{K}^x_f \cap \mathscr{K}^x_g$, and $(f^x, g^x) \cap V^x_Y = \varnothing$ implies $(f^x, g^x) \subseteq E_Y \backslash V^x_Y = \dot{\bigcup}_{\{a,b\} \in \mathscr{E}^x} (a^x, b^x)$. Hence, $(f^x, g^x) \subseteq (a^x, b^x)$ for some $\{a, b\} \in \mathscr{E}^x$ which in turn implies $\{f^x, g^x\} = [f^x, g^x] \cap V^x_Y \subseteq [a^x, b^x] \cap V^x_Y = \{a^x, b^x\}$ and thus $\{f, g\} = \{a, b\} \in \mathscr{E}^x$.

**(3.15)** For the sake of completeness let us observe that for any $\{u, v\} \in \mathscr{E}_Y$ with $\tau(u) = f$ and $\tau(v) = g$ the following statements are equivalent:

(i) $\{f, g\} \in \mathscr{E}_X$,

(iii) $\mathscr{K}_f \cap \mathscr{K}_g$ is $X$-connected,

(iii) $\mathscr{K}_f \cap \mathscr{K}_g \nsubseteq Y \times Y$,

(iii′) $\mathscr{K}_f \cap \mathscr{K}_g \neq \mathscr{K}_u \cap \mathscr{K}_v$,

(iv) $[f, g] \subseteq E_X$,

(iv′) $(f, g) \subseteq E_X^0$,

(iv″) $(f, g) \cap E_X \neq \varnothing$,

(v) $[f, g] \subseteq E^x$,

(vi) $[f, g] = \tau([u, v])$,

(vii) $(u, v) \cap V_Y^x = \varnothing$,

(vii′) $\{u, v\} \in \mathscr{E}_Y^x$,

(viii) $|f(x) - g(x)| = \|f, g\| = \|u, v\|$.

Moreover, if none of these statements holds true then there is a unique $w \in (u, v) \cap V_Y^x = (u, v) \cap U^x$ which is given explicitly by

$$\begin{aligned} w &= u + \tfrac{1}{2}(\|u, v\| + g(x) - f(x)) \cdot k_v^u \\ &= u + \tfrac{1}{2}(\|f, g\| + \|g, h_x\| - \|f, h_x\|) \cdot k_v^u \\ &= v + \tfrac{1}{2}(\|u, v\| + f(x) - g(x)) k_u^v. \end{aligned}$$

For this $w$ and its image $h = \tau(w) \in V^x \subseteq V_X$ one has

$$\begin{aligned} h(x) &= \tfrac{1}{2}(f(x) + g(x) - \|u, v\|) \\ &= \tfrac{1}{2}(\|f, h_x\| + \|h_x, g\| - \|f, g\|), \end{aligned}$$

$\{f, h\}, \{h, g\} \in \mathscr{E}_X$, $\{u, w\}, \{w, v\} \in \mathscr{E}_Y^x$,
$\|f, h\| + \|h, g\| = \|u, w\| + \|w, v\| = \|f, g\| = \|u, v\|$,
$h \notin [f, g] \neq \tau([u, v]) = [f, h] \cup [h, g] \subseteq E^x$

and

$$\begin{aligned} \mathscr{K}_h \backslash \{(x, x)\} &= (\mathscr{K}_u \cap \mathscr{K}_v) \,\dot\cup\, (\mathscr{K}_f \backslash \mathscr{P}(Y)) \,\dot\cup\, (\mathscr{K}_g \backslash \mathscr{P}(Y)) \\ &= (\mathscr{K}_f \cap \mathscr{K}_g) \,\dot\cup\, (\mathscr{K}_f \backslash \mathscr{P}(Y)) \,\dot\cup\, (\mathscr{K}_g \backslash \mathscr{P}(Y)), \end{aligned}$$

with $(x, x) \in \mathscr{K}_h$, i.e., $h = h_x$ if and only if $\|u, v\| = \|f, g\| = f(x) + g(x)$.

In particular, an edge $\{u, v\} \in \mathscr{E}_Y$ is split in at most two edges in $\mathscr{E}_Y^x$. The verification of these statements is rather easy and is left to the reader.

**(3.16)** Now let $\Gamma = (V, \mathscr{E}, l)$ be a realization of $X \subset V$ such that for any $x \in X$ there exists a subset $\mathscr{E}_x \subseteq \mathscr{E}$ such that $\Gamma_x =: (V_x =: \operatorname{supp} \mathscr{E}_x, \mathscr{E}_x, l_x = l|_{\mathscr{E}_x})$ is a realization of $Y =: X \backslash \{x\} \subset V_x$ which is isomorphic to some $\Gamma_Y^U$

via the canonical map $\varphi_{\Gamma_x} = \varphi_x\colon V_x \to P_Y$ defined by $\varphi_x(v)(y) = \overline{vy}$ $(y \in Y)$, where the distance $\overline{vy}$ is measured with respect to $\Gamma_x$. Thus $\varphi_x$ defines a bijection between $V_x$ and $V_Y \dot\cup U_x$ for some finite subset $U_x \subset E_Y^0$ and we have $\{u, v\} \in \mathscr{E}_x$ if and only if $\{\varphi_x(u), \varphi_x(v)\} \in \mathscr{E}_Y^{U_x}$ for any two elements $u, v \in V_x$ in which case one has $l(\{u, v\}) = \|\varphi_x(u), \varphi_x(v)\|$.

Let $\varphi = \varphi_\Gamma\colon V \to P_X$ denote the canonical map associated with $\Gamma$, choose a retraction $p\colon P_X \to T_X$ and let $\psi\colon V \to T_X$ denote the composition $p \circ \varphi\colon V \to P_X \to T_X$. Note that $\psi(v)(y) \leqslant \varphi(v)(y) \leqslant \varphi_x(v)(y)$ for all $x \in X$, $v \in V_x$, and $y \in Y = X\backslash\{x\}$, i.e., the functions $\psi(v)^x$ and $\varphi(v)^x$ in $P_Y$ satisfy $\psi(v)^x \leqslant \varphi(v)^x \leqslant \varphi_x(v) \in T_Y$, so (1.3) implies $\psi(v)^x = \varphi(v)^x = \varphi_x(v)$ which in turn implies $\psi(v) = \tau_x(\varphi_x(v))$ by (1.11) for all $x \in X$ and $v \in V_x$.

We claim $\|\Gamma\| \geqslant \|\Gamma_X\|$. W.l.o.g. we may assume that $\varphi_x(V_x) = V_Y \dot\cup U_x$ contains $V_Y^x = V_Y \dot\cup U^x$ (as defined above), i.e., that $U_x$ contains $U^x$ since we have $E_Y = V_Y \dot\cup U_x \dot\cup \dot\bigcup_{\{u,v\} \in \mathscr{E}_x} (\varphi_x(u), \varphi_x(v))$ and thus in case there is some $w \in V_Y^x = V_Y \dot\cup U^x$ which is not contained in $V_Y \dot\cup U_x$ there is a unique $\{u, v\} \in \mathscr{E}_x$ with $w \in (\varphi_x(u), \varphi_x(v))$. So by introducing an additional vertex $w'$ and replacing the edge $\{u, v\}$ by the two edges $\{u, w'\}$ and $\{w', v\}$ with $l(\{u, w'\}) = \|\varphi_x(u), w\|$ and $l(\{w', v\}) = \|w, \varphi_x(v)\|$ we have transformed our network $\Gamma$ into a new network of the same total length for which $w$ is contained in $\varphi_x(V_x) = V_Y \dot\cup U_x$. Note that using this procedure we have added vertices of degree 2 to $\Gamma$, only.

Now let $e = \{f, g\}$ be an edge in $\mathscr{E}_X$ and let $\mathscr{E}^e = \{\{u, v\} \in \mathscr{E} \mid \psi(u), \psi(v) \in [f, g]$ and $\psi(u) \neq \psi(v)\}$, so $\mathscr{E}^e \cap \mathscr{E}^{e'} = \varnothing$ for $e, e' \in \mathscr{E}_X$ and $e \neq e'$.

To prove our claim $\|\Gamma\| \geqslant \|\Gamma_X\|$ it is obviously enough to prove that $\|\mathscr{E}^e\| =: \sum_{\{u,v\} \in \mathscr{E}^e} l(\{u, v\}) \geqslant \|f, g\| = l(e)$ for each $e = \{f, g\} \in \mathscr{E}_X$. But the $X$-connectedness of $\mathscr{K}_f \cap \mathscr{K}_g$ implies that $X_e =: \{x \in X \mid e \in \mathscr{E}^x\} = \{x \in X \mid \mathscr{K}_f^x \cap \mathscr{K}_g^x$ is $X\backslash\{x\}$-connected$\}$ is nonempty (actually, one has $\# X_e \geqslant 2$ since $X_e$ contains the tips of any maximal tree in $\mathscr{K}_f \cap \mathscr{K}_g$), so we have $\mathscr{E}_X = \bigcup_{x \in X} \mathscr{E}^x$, and for any $x \in X$ and $e = \{f, g\} \in \mathscr{E}^x$ we have $\{f^x, g^x\} \in \mathscr{E}_Y^x = \mathscr{E}_Y^{U^x}$. So, using $U^x \subseteq U_x$ and the last remark in (3.11), we have with $\mathscr{E}_x^e =: \{\{u, v\} \in \mathscr{E}_x \mid \varphi_x(u), \varphi_x(v) \in [f^x, g^x]\}$ and $V_x^e =: \bigcup_{\{u,v\} \in \mathscr{E}_x^e} \{u, v\}$ the relations $\|\mathscr{E}_x^e\| = \sum_{\{u,v\} \in \mathscr{E}_x^e} l(\{u, v\}) = \|f^x, g^x\| = \|f, g\|$, $[f^x, g^x] = \varphi_x(V_x^e) \dot\cup \dot\bigcup_{\{u,v\} \in \mathscr{E}_x^e} (\varphi_x(u), \varphi_x(v))$ and thus—by applying $\tau_x$ and using $\psi = \tau_x \varphi_x$—$[f, g] = \psi(V_x^e) \dot\cup \dot\bigcup_{\{u,v\} \in \mathscr{E}_x^e} (\psi(u), \psi(v))$, which implies $\mathscr{E}_x^e \subseteq \mathscr{E}^e$. Thus $\|f, g\| = \|\mathscr{E}_x^e\| \leqslant \|\mathscr{E}^e\|$.

Let us note that our assumptions imply also $V_x^e = \{v \in V_x \mid \varphi_x(v) \in [f^x, g^x]\}$ for each $e = \{f, g\} \in \mathscr{E}^x$, since $\varphi_x(v) \in [f^x, g^x]$ holds by definition for $v \in V_x^e$ and $e = \{f, g\} \in \mathscr{E}^x$, whereas $v \in V_x$ and $\varphi_x(v) \in [f^x, g^x] = \varphi_x(V_x^e) \dot\cup \dot\bigcup_{\{u,v\} \in \mathscr{E}_x^e} (\varphi_x(u), \varphi_x(v))$ implies $\varphi_x(v) \in \varphi_x(V_x^e)$ in view of the partition $E_{X\backslash\{x\}} = \varphi_x(V_x) \dot\cup \dot\bigcup_{\{u,v\} \in \mathscr{E}_x} (\varphi_x(u), \varphi_x(v))$. Thus $v \in V_x^e$ because of the injectivity of $\varphi_x$.

**(3.17)** Now, continuing with the notations and assumptions of

(3.6)—including the assumption $\varphi_x(V_x) \supseteq V^x_{X\setminus\{x\}}$ or, equivalently, $U^x \subseteq U_x$—let us further assume that $\|\Gamma\| = \|\Gamma_X\|$.

It follows from the proof given in (3.16) that in this case one has $\mathscr{E} = \dot\bigcup_{e \in \mathscr{E}_X} \mathscr{E}^e$ and $\mathscr{E}^e = \mathscr{E}^e_x$ for each $x \in X$ and $e = \{f, g\} \in \mathscr{E}^x$. So, in particular, with $V^e = \bigcup_{\{u,v\} \in \mathscr{E}^e} \{u, v\}$ $(=V^e_x$ if $x \in X_e)$ one has

$$[f, g] = \psi(V^e) \,\dot\cup \dot\bigcup_{\{u,v\} \in \mathscr{E}^e} (\psi(u), \psi(v))$$

and one has $\mathscr{E} = \bigcup_{x \in X} \mathscr{E}_x$ as well as $V = \bigcup_{x \in X} V_x = \bigcup_{e \in \mathscr{E}_X} V^e$.

Note also that $\mathscr{E}_X = \bigcup_{x \in X} \mathscr{E}^x$ implies $V_X = \bigcup_{e \in \mathscr{E}_X} e = \bigcup_{x \in X} (\bigcup_{e \in \mathscr{E}^x} e) \subseteq \bigcup_{x \in X} V^x \subseteq V_X$ and $E_X = \bigcup_{\{f,g\} \in \mathscr{E}_X} [f, g] = \bigcup_{x \in X} (\bigcup_{\{f,g\} \in \mathscr{E}^x} [f, g]) \subseteq \bigcup_{x \in X} E^x \subseteq E_X$, i.e., $V_X = \bigcup_{x \in X} V^x$ and $E_X = \bigcup_{x \in X} E^x$.

We claim that our assumptions imply that $\varphi\colon V \to P_X$ maps $V$ injectively into $E_X \subseteq T_X \subseteq P_X$ and induces an isomorphism $\Gamma \simeq \Gamma^U_X$ with $U =: \varphi(V) \setminus V_X$, i.e., we claim that the following statements hold true:

(i) $\varphi$ coincides with $\psi$, i.e., $\varphi(V) \subseteq T_X$.

(ii) $V_X \subseteq \varphi(V) \subseteq E_X$,

(iii) for $\{u, v\} \in \mathscr{E}$ one has $l(\{u, v\}) = \|\varphi(u), \varphi(v)\|$,

(iv) $\varphi$ is injective,

(v) for $u, v \in V$ one has $\{u, v\} \in \mathscr{E}$ if and only if $\{\varphi(u), \varphi(v)\} \in \mathscr{E}^U_X$.

*Proof of* (i). Since $x, y \in X$, $x \neq y$, and $v \in V_x \cap V_y$ implies $\varphi(v)^x = \varphi_x(v) \in T_{X\setminus\{x\}}$ and $\varphi(v)^y = \varphi_y(v) \in T_{X\setminus\{y\}}$ and thus, using (1.2) and $X = X\setminus\{x\} \cup X\setminus\{y\}$, it implies $\varphi(v) \in T_X$ as well as $\varphi(v) = \psi(v)$, it is enough to show that $\#\{x \in X \mid v \in V_x\} \geqslant 2$ holds for any $v \in V$. But for any $v \in V$ there is some $u \in V$ and some $e \in \mathscr{E}_X$ with $\{u, v\} \in \mathscr{E}^e$ which implies $\{u, v\} \in \mathscr{E}^e_x$ and thus $v \in V_x$ for each $x \in X_e$ and we know already that $\#X_e \geqslant 2$.

*Proof of* (ii). We have $V_X = \bigcup_{x \in X} V^x$, $V = \bigcup_{x \in V} V_x$, and $V^x = \tau_x(V^x_{X\setminus\{x\}}) \subseteq \tau_x(\varphi_x(V_x)) = \psi(V_x) = \varphi(V_x) \subseteq \tau_x(E_{X\setminus\{x\}}) = E^x \subseteq E_X$ and thus

$$V_X = \bigcup_{x \in X} V^x \subseteq \bigcup_{x \in X} \varphi(V_x) = \varphi\left(\bigcup_{x \in X} V_x\right) = \varphi(V) \subseteq E_X.$$

*Proof of* (iii). This follows from $\mathscr{E} = \bigcup \mathscr{E}_x$ and $l(\{u, v\}) = \|\varphi_x(u), \varphi_x(v)\| = \|\tau_x(\varphi_x(u)), \tau_x(\varphi_x(v))\| = \|\varphi(u), \varphi(v)\|$ for $\{u, v\} \in \mathscr{E}_x$.

*Proof of* (iv). Since $\varphi|_{V_x} = \tau_x \circ \varphi_x$ is necessarily injective on $V_x$ and since $\varphi(V) \subseteq E_X = \bigcup_{x \in X} E^x$, it is obviously enough to show that $\varphi^{-1}(E^x) = V_x$, i.e., $\varphi^{-1}(E^x) \subseteq V_x$, since $\varphi(V_x) = \tau_x(\varphi_x(V_x)) \subseteq \tau_x(E_{X\setminus\{x\}}) = E^x$ holds anyway.

So assume $v \in V$ and put $h = \varphi(v)$, $X_v = \{x \in X \mid v \in V_x\}$, and $X_h =$

$\{x \in X \mid h \in E^x\}$, so we have $h \in E_X$, $\emptyset \neq X_v$, and $X_v \subseteq X_h$ since $\varphi(V_x) \subseteq E^x$. We have to show that $X_v = X_h$.

For technical reasons let us first consider the special case $\varphi(v) = h = h_y$ for some $y \in X$. Since $x \in X \backslash \{y\}$ implies $y \in X \backslash \{x\} \subseteq V_x$ we have $X_y = \{x \in X \mid y \in V_x\} \supseteq X \backslash \{y\}$ and thus $X_y \cap X_v \neq \emptyset$ since $\# X_v \geqslant 2$ by the proof of (i) for each $v \in V$. But $x \in X_y \cap X_v$, i.e., $y, v \in V_x$ and $\varphi(y) = \varphi(v)$ implies $y = v$ and thus $X \backslash \{y\} \subseteq X_y = X_v \subseteq X_h \subseteq X$. If $y \notin X_h$, we are done. Otherwise, we have $h = h_y \in V_X \cap E^y = V^y = \tau_y(V^y_{X \backslash \{y\}}) \subseteq \tau_y(\varphi_y(V_y)) = \varphi(V_y)$, i.e., $h = \varphi(w)$ for some $w \in V_y$, so—applying the above argument with respect to $w$ instead of $v$—we get $v = y = w \in V_y$, i.e., $y \in X_v$ and hence $X_v = X = X_h$. So in any case we have $X_v = X_h$ if $h = h_y$ for some $y \in X$.

From now on let us assume $h \neq h_y$, i.e., $(y, y) \notin \mathscr{K}_h$ for all $y \in X$. Consider the relation $\mathscr{L} \subseteq X_h \times X_h$ defined by

$$\mathscr{L} =: \left\{(x, y) \subseteq X_h^2 \mid h \in \bigcup_{\{f, g\} \in \mathscr{E}^x \cap \mathscr{E}^y} [f, g]\right\}.$$

Since $x \in X_v \subseteq X_h$ and $h = \varphi(v) \in [f, g]$ for some $e = \{f, g\} \in \mathscr{E}^x$ implies $\varphi_x(v) \in [f^x, g^x]$ and thus $v \in V^e_x$ by the last remark in (3.16), whereas $e \in \mathscr{E}^x \cap \mathscr{E}^y$ implies $\mathscr{E}^e_x = \mathscr{E}^e = \mathscr{E}^e_y$ and thus $V^e_x = V^e_y \subseteq V_y$, it is clear that $x \in X_v$ and $(x, y) \in \mathscr{L}$ implies $y \in X_v$, i.e., $X_v$ is a nonempty disjoint union of full connected components of $\mathscr{L}$, so we get $X_v = X_h$ once we know that $\mathscr{L}$ is $X_h$-connected which in turn follows from the following two observations:

(vi) for $h \notin \{h_x \mid x \in X\}$ one has

$$\mathscr{L} \supseteq \mathscr{L}' =: \{(x, y) \subseteq X_h^2 \mid \mathscr{K}^x_h \cap \mathscr{K}^y_h \text{ is } X \backslash \{x, y\}\text{-connected}\}, \text{ and}$$

(vii) $\mathscr{L}'$ is $X_h$-connected.

*Proof of* (vi). If $\mathscr{K}_h$ is bipartite, there is only one $\{f, g\} \in \mathscr{E}_X$ with $h \in [f, g]$ and for this $\{f, g\} \in \mathscr{E}_X$ one has $h \in (f, g)$ and thus $\mathscr{K}_h = \mathscr{K}_f \cap \mathscr{K}_g$, so we have

$$x \in X_h \Leftrightarrow \mathscr{K}^x_h \text{ is } X \backslash \{x\}\text{-connected} \Leftrightarrow \mathscr{K}^x_f \cap \mathscr{K}^x_g \text{ is } X \backslash \{x\}\text{-connected}$$
$$\Leftrightarrow \{f, g\} \in \mathscr{E}^x \qquad \text{and therefore } \mathscr{L} = X_h \times X_h \supseteq \mathscr{L}'.$$

Otherwise, one has $h \in V_X$ and thus $h \in [f, g]$ for some $\{f, g\} \in \mathscr{E}_X$ if and only if $h \in \{f, g\}$, so the set $\{e = \{f, g\} \in \mathscr{E}_X \mid h \in [f, g]\}$ corresponds to the set $\mathscr{Y}_h$ of $\mathscr{K}_h$-admissible subsets of $X$. Let $\mathscr{K} = \mathscr{K}_h$. To prove $\mathscr{L}' \subseteq \mathscr{L}$ it is enough to show that for $(x, y) \in \mathscr{L}'$ there is some $\mathscr{K}$-admissible subset $Z \subseteq X$—corresponding to an edge $e = \{h, g\} \in \mathscr{E}_X$ with $\mathscr{K} \cap \mathscr{K}_g = \mathscr{K}_Z$—such that $e \in \mathscr{E}^x \cap \mathscr{E}^y$, i.e., such that $\mathscr{K}^x_Z$ is $X \backslash \{x\}$-connected and $\mathscr{K}^y_Z$ is $X \backslash \{y\}$-

connected or, still equivalently, such that $Z \backslash \{x\}$ is $\mathscr{K}^x$-admissible (relative to $X \backslash \{x\}$) and $Z \backslash \{y\}$ is $\mathscr{K}^y$-admissible (relative to $X \backslash \{y\}$). But if $(x, y) \in \mathscr{L}'$, i.e., if $\mathscr{K}^x$ is $X \backslash \{x\}$-connected, $\mathscr{K}^y$ is $X \backslash \{y\}$-connected and $\mathscr{K}^x \cap \mathscr{K}^y$ is $X \backslash \{x, y\}$-connected, then we can find some $z \in X \backslash \{x, y\}$ with $(z, y) \in \mathscr{K}^x$ and, since $(z, z) \notin \mathscr{K}_h$, we can find some $\mathscr{K}^x \cap \mathscr{K}^y$-admissible subset $Z_0 \subseteq X \backslash \{x, y\}$ with $z \in Z_0$ which we can extend to a $\mathscr{K}^y$-admissible subset $Z_1 \subseteq X \backslash \{y\}$ with $Z_0 = Z_1 \cap (X \backslash \{x, y\}) = Z_1 \backslash \{x\}$ which in turn can be extended to a $\mathscr{K}$-admissible subset $Z \subseteq X$.

In view of the construction of $Z$ it remains to show that $Z \backslash \{x\}$ is $\mathscr{K}^x$-admissible, which would imply $(x, y) \in \mathscr{L}$. But this follows from the $X \backslash \{x, y\}$-connectivity of $(\mathscr{K}^x \cap \mathscr{K}^y)_{Z_0}$ together with $(\mathscr{K}^x \cap \mathscr{K}^y)_{Z_0} \subseteq \mathscr{K}^x_{Z \backslash \{x\}}$ and $(z, y) \in \mathscr{K}^x_{Z \backslash \{x\}}$, i.e., $z \in Z \backslash \{x\}$) and $(z, y) \in \mathscr{K}^x$.

*Proof of* (vii). By induction with respct to $\# X$ we prove the purely graphtheoretic fact that for any connected symmetric relation $\mathscr{K} \subseteq X \times X$ the relation $\bar{\mathscr{K}} \subseteq \bar{X} \times \bar{X}$, defined by $\bar{X} =: \{x \in X \mid \mathscr{K}^x$ is $X \backslash \{x\}$-connected$\}$, and $\bar{\mathscr{K}} = \{(x, y) \in \bar{X}_2 \mid \mathscr{K}^x \cap \mathscr{K}^y$ is $X \backslash \{x, y\}$-connected$\}$, is itself connected.

Since an element $x \in X$ is in $\bar{X}$ if and only if it is a tip of a maximal tree in $\mathscr{K}$ and since any maximal tree has at least two tips, we see that $\bar{X} = \operatorname{supp} \bar{\mathscr{K}}$. Thus it is enough to show that for any $x \in \bar{X}$ the relation $\bar{\mathscr{K}}^x$ is $\bar{X} \backslash \{x\}$-connected. But this follows directly from our induction hypothesis since $\bar{\mathscr{K}}^x$ is easily seen to coincide with $\bar{\mathscr{K}}_1 \subseteq \bar{X}_1 \times \bar{X}_1$, if $X_1 =: X \backslash \{x\}$ and $\mathscr{K}_1 = \mathscr{K}^x \cap \{(y, z) \in X_1^2 \mid (y, x), (x, z) \in \mathscr{K}\}$.

*Proof of* (v). Let us first observe that for each $\{f, g\} \in \mathscr{E}_X$ one has $(f, g) \cap U = (f, g) \cap \varphi(V) = (f, g) \cap \bigcup_{e \in \mathscr{E}_X} \varphi(V^e) = (f, g) \cap \varphi(V^{\{f, g\}})$, since for $e = \{f', g'\} \neq \{f, g\}$ one has $(f, g) \cap \varphi(V^e) \subseteq (f, g) \cap [f', g'] = \varnothing$. The rest follows now from the injectivity of $\varphi$ and the comparison of the two partitions

$$E_X \backslash V_X^U = \dot{\bigcup_{\{f, g\} \in \mathscr{E}_X^U}} (f, g)$$

and

$$\begin{aligned} E_X \backslash V_X^U = (E_X \backslash V_X) \backslash U &= \Big( \dot{\bigcup_{\{f, g\} \in \mathscr{E}_X}} (f, g) \Big) \Big\backslash U \\ &= \dot{\bigcup_{\{f, g\} \in \mathscr{E}_X}} ((f, g) \backslash (U \cap (f, g))) \\ &= \dot{\bigcup_{\{f, g\} \in \mathscr{E}_X}} ((f, g) \backslash \varphi(V^{\{f, g\}})) \\ &= \dot{\bigcup_{e = \{f, g\} \in \mathscr{E}_X}} \; \dot{\bigcup_{\{u, v\} \in \mathscr{E}^e}} (\varphi(u), \varphi(v)) \\ &= \dot{\bigcup_{\{u, v\} \in \mathscr{E}}} (\varphi(u), \varphi(v)), \end{aligned}$$

since these two partitions of $E_X \setminus V_X^U$ into disjoint open connected subsets must necessarily coincide, so for $u, v \in V$ we have "$\{u, v\} \in \mathscr{E} \Rightarrow (\varphi(u), \varphi(v)) = (f, g)$ for some $\{f, g\} \in \mathscr{E}_X^U \Rightarrow \{\varphi(u), \varphi(v)\} = \{f, g\} \in \mathscr{E}_X^U$," as well as "$\{\varphi(u), \varphi(v)\} \in \mathscr{E}_X^U \Rightarrow (\varphi(u), \varphi(v)) = (\varphi(u'), \varphi(v'))$ for some $\{u', v'\} \in \mathscr{E} \Rightarrow \{\varphi(u), \varphi(v)\} = \{\varphi(u'), \varphi(v')\}$ for some $\{u', v'\} \in \mathscr{E} \Rightarrow \{u, v\} = \{u', v'\} \in \mathscr{E}$."

**(3.18)** Finally assume $\Gamma = (V, \mathscr{E}, l)$ to fulfil all of the above conditions except perhaps $\varphi_x(V_x) \supseteq V_{X \setminus \{x\}}^x$. By (3.16) we know that we can construct a new network $\Gamma' = (V', \mathscr{E}', l')$ with $V \subseteq V'$, $\|\Gamma\| = \|\Gamma'\|$, and $\varphi_x'(V_x') \supseteq V_{X\setminus\{x\}}^x$ such that $V' \setminus V$ consists of vertices of degree 2 only and such that $\varphi_x'(V_x' \setminus V) \subseteq V_{X\setminus\{x\}}^x$.

We may apply our results with respect to $\Gamma'$ to conclude that $\varphi' \colon V' \to E_X$ is injective, contains $V_X$ in its image, induces an isomorphism $\Gamma' \to \Gamma_X^{U'}$ with $U' = \varphi'(V') \setminus V_X$, and satisfies $\varphi'(V' \setminus V) = \bigcup_{x \in X} \varphi'(V_x' \setminus V) = \bigcup_{x \in X} \tau_x \varphi_x'(V_x' \setminus V) \subseteq \bigcup_{x \in X} \tau_x(V_{X\setminus\{x\}}^x) \subseteq V_X$. So for each $v' \in V' \setminus V$ the degree of $\varphi'(v') \in V_X$ must be 2 which implies $\varphi'(v') \in \{h_x \mid x \in X\}$, since $\# N_f \geqslant 3$ for each $f \in V_X \setminus X$. But $\varphi'(v') = h_x = \varphi'(x)$ for some $x \in X$ implies $v' = x \in X \subset V$. So we have $V = V'$ and thus $\Gamma = \Gamma'$ already satisfies $\varphi_x(V_x) \supseteq V_{X\setminus\{x\}}^x$ for any $x \in X$.

**(3.19)** It is now easy to prove Theorem 7′ by induction with respect to $\# X$: in case $\# X = 2$ the verification of Theorem 7′ is trivial. So consider the case $\# X > 2$, assume Theorem 7′ to be true for all spaces $X \setminus \{x\}$ $(x \in X)$, and let $\Gamma$ be a realization of $X$ which like $\Gamma_X$ "contains" a hereditarily optimal realization of each $X \setminus \{x\}$ $(x \in X)$. From (3.16) we conclude $\|\Gamma\| \geqslant \|\Gamma_X\|$, so $\Gamma_X$ is a least necessarily a hereditarily optimal realization of $X$. Moreover if $\Gamma$ is hereditarily optimal, too, we get $\|\Gamma\| = \|\Gamma_X\|$ and thus, using (3.17) and (3.18) we conclude that $\Gamma$ is canonically isomorphic to some $\Gamma_X^U$. So altogether we have proved Theorem 7′ for $X$.

**(3.20)** I conjecture that for any finite metric space $X$ there is a subset $\mathscr{E}_0 \subseteq \mathscr{E}_X$ with $X \subseteq V_0 =: \operatorname{supp} \mathscr{E}_0$ such that $\Gamma_0 = (V_0, \mathscr{E}_0, l_X|_{\mathscr{E}_0})$ is an optimal realization of $X$ and that any optimal realization of $X$ is essentially isomorphic to some such $\Gamma_0$. Moreover, I conjecture that for an open dense subset of the set of all metrics on $X$ there is only one possible choice of $\mathscr{E}_0$.

## 4. Trees

**(4.1)** To prepare the proof of Theorem 8 let us note at first that a metric space $X$ is tree-like if and only if $T_X$ is tree-like: since subspaces of tree like spaces are obviously tree-like and since for $f_1, f_2, f_3, f_4 \in T_X$ and

$X_i = X \cup \{f_1, ..., f_{i-1}\}$ $(i = 1, 2, 3, 4)$ we have natural identifications $T_X = T_{X_1} = T_{X_2} = T_{X_3} = T_{X_4}$ and thus $f_i \in T_{X_i}$ as well as $X_{i+1} = X_i \cup \{f_i\}$ $(i = 1, 2, 3)$ it will be enough to show that $X \cup \{f\}$ is tree-like whenever $X$ is tree-like and $f \in T_X$.

But for $x, y, v \in X$ and $f \in T_X$ the tree-likeness of $X$ (in form of the condition (T′)) implies $xy + \|h_v, f\| = xy + f(v) = \sup(xy + vw - f(w) \mid w \in X) \leqslant \sup(xv + yw - f(w), xw + yv - f(w) \mid w \in X) = \sup(xv + f(y), yv + f(x)) = \sup(xv + \|h_y, f\|, yv + \|h_x, f\|)$ and thus the tree-likeness of $X \cup \{f\}$.

Note also that a space $X$ is tree-like if and only if the conditions (T) or (T′) are fulfilled for all $x, y, v, w \in X$ with $\#\{x, y, v, w\} = 4$, since they hold for $\#\{x, y, v, w\} < 4$ quite trivially in any metric space. Thus, using (1.16), a space $X$ is tree-like if and only if $T_Y$ is one dimensional for all $Y \subseteq X$ with $\#Y = 4$.

**(4.2)** Next let us define for a metric space $X$ and two elements $x, y \in X$ the subset

$$\langle x, y \rangle = \langle x, y \rangle_X =: \{z \in X \mid xz + zy = xy\}$$

which is always a closed subset containing $x$ and $y$.

The following statements are more or less obvious:

(a) $z \in \langle x, y \rangle$ implies $\langle x, z \rangle \cap \langle z, y \rangle = \{z\}$ and $\langle x, z \rangle \cup \langle z, y \rangle \subseteq \langle x, y \rangle$.

(b) For any isometry $\varphi$: $[0, t] \hookrightarrow X$ with $\varphi(0) = x$ and $\varphi(t) = y$ one has $\varphi([0, t]) \subseteq \langle x, y \rangle$ and $h_x \cdot \varphi = h_x|_{\langle x, y \rangle} \cdot \varphi = \mathrm{Id}_{[0,t]}$.

(c) For $x, y, v, w \in X$ the following conditions are equivalent:

(i) $xv + vw + wy = xy$,

(ii) $v \in \langle x, w \rangle$ and $w \in \langle x, y \rangle$,

(iii) $v \in \langle x, y \rangle$ and $w \in \langle v, y \rangle$,

(iv) $\langle v, w \rangle \subseteq \langle x, w \rangle \cap \langle v, y \rangle \subseteq \langle x, y \rangle$.

(d) $h_x|_{\langle x, y \rangle} \to [0, xy] : z \mapsto zx$ is an isometric embedding if and only if for all $v, w \in \langle x, y \rangle$ one has $xv + vw + wy = xy$ or $xw + wv + vy = xy$.

If this holds for all $x, y \in X$, the space $X$ will be called thready.

**(4.3)** A space $X$ satisfies the condition (T1), considered in the introduction, if and only if the map

$$h_x|_{\langle x, y \rangle} : \langle x, y \rangle \to [0, xy] : z \mapsto zx$$

is a bijective isometry for all $x, y \in X$ or, equivalently, if and only if $X$ is thready and $\langle x, y\rangle$ is connected for all $x, y \in X$.

So, in particular, for any three elements $x, y, z$ in such a space $X$ there is a unique $u = u^x_{y,z} \in X$ with

$$\langle x, y\rangle \cap \langle x, z\rangle = \langle x, u\rangle,$$

namely the unique element $u \in \langle x, y\rangle \cap \langle x, z\rangle$ with $ux \geqslant vx$ for all $v \in \langle x, y\rangle \cap \langle x, z\rangle$ or, equivalently, with $\langle u, y\rangle \cap \langle u, z\rangle = \{u\}$.

*Proof.* Assume $X$ to satisfy (T1). It is enough to show that $z \in \langle x, y\rangle \subseteq X$ implies $\varphi_{x,y}(xz) = z$, i.e., that $\varphi_{x,y}$—being a right inverse of $h_x|_{\langle x,y\rangle}$ in view of (4.2b)—is also a left inverse of $h_x|_{\langle x,y\rangle}$. But if $z \in \langle x, y\rangle$ and if $\psi$: $[0, xy] \to \langle x, z\rangle \cup \langle z, y\rangle \subseteq \langle x, y\rangle$ is defined by

$$\begin{aligned} \psi(t) &= \varphi_{x,z}(t) && \text{for} \quad 0 \leqslant t \leqslant xz, \\ &= \varphi_{z,y}(t - xz) && \text{for} \quad xz \leqslant t \leqslant xy, \end{aligned}$$

then $\psi$ satisfies $\psi(0) = x$, $\psi(xz) = t$, $\psi(xy) = y$, and $\psi(t_1)\,\psi(t_2) = |t_1 - t_2|$ for $t_1, t_2 \in [0, xz]$ or $t_1, t_2 \in [xz, xy]$, as well as $\psi(t_1)\,\psi(t_2) \leqslant \psi(t_1)\,\psi(xz) + \psi(xz)\,\psi(t_2) = (xz - t_1) + (t_2 - xz) = t_2 - t_1 = |t_2 - t_1|$ for $0 \leqslant t_1 \leqslant xz \leqslant t_2 \leqslant xy$, which together with $xy \leqslant x\psi(t_1) + \psi(t_1)\,\psi(t_2) + \psi(t_2)\,y \leqslant t_1 + \psi(t_1)\,\psi(t_2) + (xy - t_2)$, that is, $t_2 - t_1 \leqslant \psi(t_1)\,\psi(t_2)$ implies $\psi(t_1)\,\psi(t_2) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, xy]$ and thus $\psi = \varphi_{x,y}$ which in turn implies $\varphi_{x,y}(xz) = \psi(xz) = z$.

Vice versa, if $h_x|_{\langle x,y\rangle}$: $\langle x, y\rangle \simeq [0, xy]$ is a bijective isometry, then its inverse $\varphi = \varphi_{x,y}$: $[0, xy] \simeq \langle x, y\rangle \subseteq X$ is an isometry $\varphi$ with $\varphi(0) = x$ and $\varphi(xy) = y$ and it is the only such isometry in view of (4.2b): if $\varphi'$: $[0, xy] \to X$ is another isometry with $\varphi'(0) = x$ and $\varphi'(xy) = y$, then $h_x|_{\langle x,y\rangle} \circ \varphi = h_x|_{\langle x,y\rangle} \circ \varphi'$ together with the injectivity of $h_x|_{\langle x,y\rangle}$ implies $\varphi = \varphi'$.

**(4.4)** If $X$ is a tree (as defined in the introduction), then $X$ is "median" (cf. [20]), i.e., one has $\langle x, y\rangle \cap \langle y, z\rangle \cap \langle z, x\rangle \neq \varnothing$ for all $x, y, z \in X$.

*Proof.* Assume $X$ to be a tree and $x, y, z \in X$. W.l.o.g. we may assume $y \notin \langle x, z\rangle$ and $z \notin \langle x, y\rangle$. Consider the element $u = u^x_{y,z}$ with $\langle x, y\rangle \cap \langle x, z\rangle = \langle x, u\rangle$ and the map $\varphi$: $[0, 1] \to X$ defined by

$$\begin{aligned} \varphi(t) &= \varphi_{x,y}(xy - 2t \cdot yu) && \text{for} \quad 0 \leqslant t \leqslant \tfrac{1}{2}, \\ &= \varphi_{x,z}(xu + (2t - 1) \cdot uz) && \text{for} \quad \tfrac{1}{2} \leqslant t \leqslant 1. \end{aligned}$$

Since $\varphi_{x,y}(xy - yu) = \varphi_{x,y}(xu) = u = \varphi_{x,z}(xu)$, the map $\varphi$ is well defined and continuous. Since $y \neq u \neq z$ and $\varphi([0, \frac{1}{2}]) \cap \varphi([\frac{1}{2}, 1]) = \varphi_{x,y}([xu, xy]) \cap \varphi_{x,z}([xu, xz]) \subseteq \langle u, y \rangle \cap \langle u, z \rangle = \{u\}$, it is injective. Thus, using (T2), we have $u = \varphi(\frac{1}{2}) \in \langle \varphi(0), \varphi(1) \rangle = \langle y, z \rangle$ and therefore $u \in \langle x, y \rangle \cap \langle x, z \rangle \cap \langle y, z \rangle \neq \varnothing$.

**(4.5)** If a metric space $X$ is tree-like, then it is thready. Vice versa, if $X$ is thready and median, then it is tree-like and one has $\langle x, z \rangle \subseteq \langle x, y \rangle \cup \langle y, z \rangle$ for all $x, y, z \in X$.

*Proof.* Assume $X$ to be tree-like, $x, y \in X$, and $v, w \in \langle x, y \rangle$. W.l.o.g. assume $xv \leqslant xw$. This implies $xv + yw \leqslant xw + wy = xy \leqslant xy + vw$ as well as $xv + yw \leqslant xw + yv$ and so, in view of (T), it implies $xw + yv = xy + vw = xv + vy + vw$, i.e., $xw = xv + vw$ and therefore $xv + vw + wy = xw + wy = xy$. So $X$ is thready.

Now assume $X$ to be thready and median and assume $x, y, v, w \in X$. Choose some $a \in \langle x, v \rangle \cap \langle v, w \rangle \cap \langle w, x \rangle$ and some $b \in \langle y, v \rangle \cap \langle v, w \rangle \cap \langle w, y \rangle$. Since $a, b \in \langle v, w \rangle$ and $X$ is supposed to be thready we have $ab + bw = aw$ or $ba + aw = bw$ and therefore $xy + vw \leqslant (xa + ab + by) + (vb + bw) = (xa + (ab + bw)) + (yb + bv) = xw + yv$ or $xy + vw \leqslant (xa + ab + by) + (va + aw) = (xa + av) + ((yb + (ba + aw)) = xv + yw$, i.e., $xy + vw \leqslant \sup(xw + yv, xv + yw)$, so $X$ is tree-like.

The last remark follows from the fact that after choosing some $a \in \langle x, y \rangle \cap \langle y, z \rangle \cap \langle z, x \rangle$ one has $\langle x, z \rangle = \langle x, a \rangle \cup \langle a, z \rangle \subseteq \langle x, y \rangle \cup \langle y, z \rangle$.

**(4.6)** Let us now consider for a metric space $X$ and two elements $x, y \in X$ the map $h_{x,y}\colon X \to [0, xy] \subseteq \mathbb{R}\colon v \mapsto \frac{1}{2}(xy + xv - yv)$. We claim, that for a tree-like metric space $X$, any two elements $x, y \in X$ and any $r \in [0, xy]$ the set $h_{x,y}^{-1}(r) \cap \langle x, y \rangle = h_x^{-1}(r) \cap \langle x, y \rangle$ consists of at most one element and that $h_{x,y}^{-1}(r) \backslash \langle x, y \rangle$ is an open subset of $X$.

*Proof.* The first statement follows from $h_{x,y}|_{\langle x,y \rangle} = h_x|_{\langle x,y \rangle}$ and (4.5). To prove the second statement assume $v \in h_{x,y}^{-1}(r) \backslash \langle x, y \rangle$ and put $\varepsilon = \frac{1}{2}(xv + vy - xy)$. Since $v \notin \langle x, y \rangle$ we have $\varepsilon > 0$. Now assume $w \in X$ and $vw < \varepsilon$. We claim that $w \in h_{x,y}^{-1}(r) \backslash \langle x, y \rangle$: $w \notin \langle x, y \rangle$ follows from $xw + wy \geqslant xv - vw + yv - vw = xy + 2\varepsilon - 2vw > xy$, and $w \in h_{x,y}^{-1}(r)$, being equivalent to $xw - yw = xv - yv$, i.e., $xw + yv = xv + yw$, follows from the tree-likeness of $X$ and $xy + vw = xv + vy - 2\varepsilon + vw < xv + vy - vw \leqslant xv + yw$.

Note that the openness of $h_{xy}^{-1}(r) \backslash \langle x, y \rangle$ implies that the small inductive dimension $\operatorname{ind} X$ of any tree-like space $X$ (see [21] for the definition of it) is smaller than or equal to 1, because for $\varepsilon, \eta > 0$ and $x, y \in X$ with $xy = \varepsilon$ one has $h_y^{-1}(\varepsilon) \cap h_x^{-1}(\eta) = h_y^{-1}(\varepsilon) \cap (h_{x,y}^{-1}(\eta/2) \backslash \langle x, y \rangle)$, so for any $x \in h_y^{-1}(\varepsilon)$ the canonical neighbourhood system $U_x(\eta, h_y^{-1}(\varepsilon)) = \{z \in h_y^{-1}(\varepsilon) \mid xz < \eta\}$

($\eta > 0$) of $x$ in $h_y^{-1}(\varepsilon)$ consists of subsets of $h_y^{-1}(\varepsilon)$ which are simultaneously closed and open in $h_y^{-1}(\varepsilon)$. So in particular, $\dim X \leqslant 1$ for any separable tree-like space.

**(4.7)** It is now easy to prove that a metric space $X$ is a tree if and only if it is tree-like and connected if and only if it is thready, median, and connected.

*Proof.* If $X$ is a tree, then $X$ is obviously connected, thready, and median by (4.3) and (4.4). If $X$ is thready and median, then it is tree-like by (4.5). Finally, if $X$ is tree-like and connected, then it is thready by (4.5) and the map $h_x|_{\langle x, y\rangle}: \langle x, y\rangle \to [0, xy]$ must be surjective, since otherwise there would exist some $r \in [0, xy]$ with $h_x^{-1}(r) \cap \langle x, y\rangle = h_{x,y}^{-1}(r) \cap \langle x, y\rangle = \varnothing$ which implies $r \in (0, xy)$ and therefore $x \in U_1 = \{z \in X \mid h_{x,y}(z) < r\}$ and $y \in U_2 = \{z \in X \mid h_{x,y}(z) > r\}$. Thus $X = U_1 \cup U_2 \cup h_{x,y}^{-1}(r) = U_1 \cup U_2 \cup (h_{x,y}^{-1}(r)\backslash \langle x, y\rangle)$ would be a partition of $X$ into three disjoint open sets, two of which are nonempty (at least), contradicting the connectedness of $X$. So a tree-like connected space $X$ necessarily satisfies (T1) by (4.3). It satisfies also (T2), since for any continuous map $\varphi: [0, 1] \to X$ with $\varphi(0) = x$ and $\varphi(1) = y$ and for any connected component $(t_1, t_2)$ of the open subset $\varphi^{-1}(X\backslash\langle x, y\rangle) \subseteq (0, 1)$ one has $\varphi(t_1) = \varphi(t_2)$—so, for an injective continuous map $\varphi$: $[0, 1] \to X$ one has necessarily $\varphi([0, 1]) \subseteq \langle \varphi(0), \varphi(1)\rangle$—because $X\backslash\langle x, y\rangle$ is a disjoint union of the open sets $h_{x,y}^{-1}(r)\backslash\langle x, y\rangle$ ($r \in [0, xy]$) by (4.6). So the connected set $\varphi((t_1, t_2)) \subseteq X\backslash\langle x, y\rangle$ must necessarily be contained in one such open set, i.e., there is some $r \in [0, xy]$ with $h_{x,y}(\varphi(t)) = \frac{1}{2}(xy + x\varphi(t) - y\varphi(t)) = r$ for all $t \in (t_1, t_2)$ which implies $h_{x,y}(\varphi(t_1)) = h_{x,y}(\varphi(t_2)) = r$. Since $\varphi(t_1), \varphi(t_2) \notin X\backslash\langle x, y\rangle$, i.e., $\varphi(t_1), \varphi(t_2) \in \langle x, y\rangle$, this implies $\varphi(t_1) = \varphi(t_2)$ by (4.6).

**(4.8)** We are now ready to prove Theorem 8. Since $T_X$ is connected for any $X$, the implication (i) $\Rightarrow$ (ii) follows from (4.1) and (4.7). The implication (ii) $\Rightarrow$ (vi) follows from the fact that $x, y, z \in X$, $f(x) + f(y) > xy$, and $f(y) + f(z) > yz$ implies $f \notin \langle x, y\rangle \cup \langle y, z\rangle$ and thus $f \notin \langle x, z\rangle \subseteq \langle x, y\rangle \cup \langle y, z\rangle$, i.e., $f(x) + f(z) > xz$. In particular, one has $T_X^0 = \{f \in T_X \mid \operatorname{supp} \mathscr{K}_f = X\} = \bigcup_{x,y\in X} \langle x, y\rangle_{T_X}$, so we see that $T_X^0 \subseteq T_X$ is tree-like and connected and thus a tree, whenever $T_X$ is a tree, i.e., we have (ii) $\Rightarrow$ (iii). The implications (iii) $\Rightarrow$ (vii) and (iv) $\Rightarrow$ (iv$'$) $\Rightarrow$ (v) are trivial; (vii) $\Rightarrow$ (i) follows from (4.7), (v) $\Rightarrow$ (i) follows from (1.16), and (i) $\Rightarrow$ (iv) follows from the last remark in (4.6). The remaining implication (vi) $\Rightarrow$ (i) can also be deduced from (1.16), but it follows also from the proof of the following description of finite tree-like spaces.

**(4.9)** If $X$ is finite, then the following conditions are equivalent:

(i) $X$ is tree-like;

(vi) $\mathscr{K}_f$ is completely multipartite for any $f \in T_X$;

(viii) $T_X = E_X$;

(ix) the graph $(V_X, \mathscr{E}_X)$ is a tree (in the usual sense of graph theory).

*Proof.* (vi) ⇒ (viii). Since $\#X < \infty$ implies $\mathscr{K}_f \neq \varnothing$ (even $\operatorname{supp} \mathscr{K}_f = X$) we see that $\mathscr{K}_f$—being nonempty and multipartite—is necessarily connected for any $f \in T_X$, so we have $f \in E_X$ if $f \in T_X \supseteq E_X$, i.e., we have $E_X = T_X$.

Note that $T_X = E_X$ and $E_X = V_X \cup \dot{\bigcup}_{\{f,g\} \in \mathscr{E}_X} (f, g) = \bigcup_{\{f,g\} \in \mathscr{E}_X} [f, g]$ implies once again that $\dim T_X = 1$, so (viii) implies (i).

(viii) ⇒ (ix). Since $E_X = T_X$ is a topological realization of $(V_X, \mathscr{E}_X)$ and since $T_X$ is contractible, $(V_X, \mathscr{E}_X)$ must be a tree.

(ix) ⇒ (i). This holds at least for $\#X = 4$ in view of (1.16), so it holds in general since for any $Y \subseteq X$ there exists a "subdivision" of $(V_Y, \mathscr{E}_Y)$ which is isomorphic to a subgraph of $(V_X, \mathscr{E}_X)$ by (3.14).

**(4.10)** To conclude the proof of Theorem 8 let $\varphi: X \to T$ be an isometric embedding of $X$ into a tree $T$. By (1.11) we can extend it to an isometric embedding $\psi$ of $T_X$ into $T_T$ which necessarily maps $\langle f, g \rangle_{T_X}$ bijectively onto $\langle \psi(f), \psi(g) \rangle_{T_T}$ for all $f, g \in T_X$ and so it maps $T_X^0 = \bigcup_{x,y \in X} \langle h_x, h_y \rangle_{T_X}$ onto $\bigcup_{x,y \in X} \langle h_{\varphi(x)}, h_{\varphi(y)} \rangle_{T_T} \subseteq \bigcup_{u,v \in T} \langle h_u, h_v \rangle_{T_T} = \bigcup_{u,v \in T} \{h_w \mid w \in \langle u, v \rangle_T\} \subseteq \{h_w \mid w \in T\} = T$. So we have at least one isometric extension $\psi|_{T_X^0}: T_X^0 \to T$ of $\varphi: X \to T$. That this extension is unique, is obvious in view of (4.3) since $f \in T_X^0$ and, say, $f(x) + f(y) = xy$ for some $x, y \in X$ implies $\psi'(f) \in \langle \varphi(x), \varphi(y) \rangle \cap h_{\varphi(x)}^{-1}(f(x)) = \{\psi(f)\}$ for any extension $\psi': T_X^0 \to T$ of $\varphi$.

Finally, we have seen already that a metric space is a tree if and only if it is connected and tree-like, so it remains to show that the completion $\bar{X}$ of a tree $X$ coincides with $T_X$. We have observed already in the Introduction, that $\bar{X}$ is always contained in $T_X$. To show that $\bar{X} = T_X$ assume $f \in T_X$ and fix some $x \in X$. Then for each $n \in \mathbb{N}$ there is some $x_n \in X$ with $f(x) + f(x_n) \leqslant xx_n + (1/n)$. Since $T_X$ is a tree and thus it is median, there is some $g_n \in T_X$ with $g_n \in \langle x, x_n \rangle_{T_X} \cap \langle x, f \rangle_{T_X} \cap \langle f, x_n \rangle_{T_X}$ which implies $\| g_n, f \| = \frac{1}{2}(f(x) + f(x_n) - xx_n) < 1/2n$, as well as $g_n \in \langle x, x_n \rangle_{T_X} = \langle x, x_n \rangle_X$ (since $X$ is supposed to be a tree). So we have $g_n = h_{y_n}$ for some $y_n \in X$ and $f(y_n) = \| g_n, f \| < 1/2n$ which implies $f = \lim_n y_n \in \bar{X}$.

(4.11) *Remark.* (a) It should be kept in mind that metric trees can be considerably more complicated than graph theoretic trees, e.g., the space $X = \{x = (x_n)_{n \in \mathbb{R}} \mid x_n \in \mathbb{R}, \sum_n |x_n| < \infty\}$ is tree-like if the metric on $X$ is defined in the following way: for $x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in X$ define $m(x, y) =: \inf(n \in \mathbb{N} \mid x_n \neq y_n)$ and if $m = m(x, y)$ put

$$xy =: |x_m - y_m| + \sum_{n > m} (|x_n| + |y_n|).$$

Note that for $x, y, z \in X$ one has $m(x, y) = \max(m(x, y), m(x, z), m(y, z))$ if and only if $m(x, z) = m(y, z)$. Thus if $x, y, v, w \in X$ and $\#\{x, y, v, w\} = 4$ and if, say, $m = \max(m(a, b) \mid a, b \in \{x, y, v, w\}, a \neq b) = m(x, y)$ we have either $m(x, y) = m(a, b)$ for all $a, b \in \{x, y, v, w\}$ with $a \neq b$, in which case we may assume w.l.o.g. that $x_m < y_m < v_m < w_m$ which implies easily $xv + yw = xw + yv > xy + vw$, or we have, say, $m = m(x, y) = m(x, v) = m(y, v) > k = m(x, w) = m(y, w) = m(v, w)$ in which case we may assume w.l.o.g. that either $x_m < y_m < v_m$ and $y_m \leqslant 0$ or $v_m < y_m < x_m$ and $y_m \geqslant 0$. So in both cases we have $|x_m| + |y_m - v_m| = |y_m| + |x_m - v_m|$ which implies again $xv + yw = xw + yv > xy + vw$, or we have, say, $m = m(x, y) > l = m(x, v) = m(y, v)$ and $m > k = m(x, w) = m(y, w)$ which again implies $xv + yw = xw + yv > xy + vw$.

One verifies easily that $X$ is complete with respect to this metric and that it is the completition of $X_0 = \{x \in X \mid x_n = 0$ for almost all $n \in \mathbb{N}\}$. Moreover, $X_0$ is obviously connected and, thus, it is a tree, too. We will study these trees as well as the subtrees

$$X_0^k = \{(x_0, x_1, ..., x_n, ...) \in X_0 \mid x_i \geqslant 0 \text{ for all } i \geqslant 1,$$
$$x_i = 0 \text{ if } x_{i-1} = \cdots = x_{i-k} = 0 \text{ for all } i \geqslant k+1\}$$

($k \in \mathbb{N}$) specifically in a later paper, since they have rather interesting universal properties and, in particular, they are highly homogeneous, so it may be of interest to look at their automorphism groups.

(b) It follows easily from (4.9) and from Section 3 that for a finite tree-like space $X$ the network $\Gamma_X = (V_X, \mathscr{E}_X, l_X)$ is not only an hereditarily optimal, but is also an optimal realization of $X$ and that any optimal realization of $X$ is essentially isomorphic to $T_X$ (cf. [28, 10, 22, 31]).

(c) For reconstructing phylogenetic trees from data concerning present species which data can be represented generally in form of a metric $D$ defined on the set $X$ of those present species which are to be studied it seems to be interesting to look for algorithms by which one can find for any metric $D$ defined on a finite set $X$ a metric $D'$, also defined on $X$, such that $(X, D')$ is tree-like, $D'(x, y) \geqslant D(x, y)$ for all $x, y \in X$, and $\sum_{x,y \in X} (D'(x, y) - D(x, y))$ or $\sum_{x,y \in X} (D'(x, y) - D(x, y))^2$ is as small as possible.

First attempts in this direction have been made already and it is hoped that this way one can replace the rather coarse-grained cluster analysis, generally used for the mathematical reconstruction of phylogenetic trees, by a much more refined method (cf. [1]).

(d) In this context it seems also worthwile to observe that for a tree-like metric space $X$ one has $T_X = T_X^0$ at least if $X$ is compact or if the set $\{xy + yz - xz \mid x, y, z \in X\}$ is a discrete subset of $\mathbb{R}$, e.g., if

$\{xy \mid x, y \in X\} \subseteq \mathbb{N}$: if $X$ is compact and $x \in X$, one has $f(x) = \max(xy - f(y) \mid y \in X)$ for each $f \in T_X$ and thus $f(x) + f(y) = xy$ for some $y \in X$. If $\{xy + yz - xz \mid x, y, z \in X\}$ is discrete, $x \in X$, and $f \in T_X$ then one can find elements $x_i \in X$ ($i \in \mathbb{N}$) such that $\varepsilon_i =: f(x) + f(x_i) - xx_i$ converges monotonically decreasing towards 0. We claim that $\varepsilon_i = 0$ for some $i \in \mathbb{N}$. Otherwise we may assume $0 < \varepsilon_j < \varepsilon_i$ for $i < j$ which implies $xx_j + f(x_i) = f(x) + f(x_j) - \varepsilon_j + f(x_i) > f(x) + f(x_j) - \varepsilon_i + f(x_j) = xx_i + f(x_j)$ and thus $xx_j + f(x_i) = x_j x_i + f(x)$ by (4.1). But this in turn implies $2f(x) - \varepsilon_i = (xx_j + f(x_i) - x_j x_i) + (xx_i - f(x_i)) = xx_j + xx_i - x_i x_j \in \{ab + bc - ac \mid a, b, c \in X\}$, contradicting the discreteness of this set.

The subset $Z = \{((1/i), i, 0, 0, \ldots) \mid i = 1, 2, \ldots\} \subseteq X_0 \subseteq X$ of the space $X_0$ considered in (a) shows, that it is not enough to assume the set of distances itself to be discrete since $\{xy \mid x, y \in Z\} = \{i + j + (1/i) - (1/j) \mid i, j \in \mathbb{N}, 1 \leqslant i \lneqq j\}$ is certainly discrete and $f\colon Z \to \mathbb{R} : ((1/i), i, 0, 0, \ldots) \mapsto i + (1/i)$ is in $T_Z$, but not in $T_Z^0$: for $1 \leqslant i \lneqq j$ and $i, j \in \mathbb{N}$ one has $f(x_i) + f(x_j) - x_i x_j = 2/j$.

## 5. The Combinatorial Dimension of Metric Spaces

**(5.1)** Let us start with the following observation which is a simple consequence of the Hahn–Banach theorem: if $X$ and $A$ are sets and if $\rho\colon A \to \mathbb{R}$ and $\sigma\colon A \times X \to \mathbb{R}$ are maps such that for each $\alpha \in A$ the map $\sigma_\alpha\colon X \to \mathbb{R} : x \mapsto \sigma(\alpha, x)$ has finite support (i.e., $\operatorname{supp}(\sigma_\alpha) =: \{x \in X \mid \sigma(\alpha, x) \neq 0\}$ is finite for all $\alpha \in A$) then there exists a map $f\colon X \to \mathbb{R}$ with $\sum_{x \in X} \sigma(\alpha, x) f(x) \leqslant \rho(\alpha)$ for all $\alpha \in A$ if and only if for each $h\colon X \to \mathbb{R}$ of finite support one has $-\infty < i(h) =: \inf(\sum_\alpha \tau(\alpha) \rho(\alpha) \mid \tau \in \mathbb{R}_h^A)$ with $\mathbb{R}_h^A$ denoting the set of all nonnegative maps $\tau\colon A \to \mathbb{R}$ of finite support with $\sum_\alpha \tau(\alpha) \sigma(\alpha, x) = h(x)$ for all $x \in X$. In particular, if $\mathbb{R}_h^A \neq \varnothing$ for all $h\colon X \to \mathbb{R}$ of finite support, then there exists such a map $f\colon X \to \mathbb{R}$ with $\sum_x \sigma(\alpha, x) f(x) \leqslant \rho(\alpha)$ for all $\alpha \in A$ if and only if $i(0) = 0$, i.e., if and only if $\sum_\alpha \tau(\alpha) \rho(\alpha) \geqslant 0$ for all nonnegative maps $\tau\colon A \to \mathbb{R}$ of finite support with $\sum_\alpha \tau(\alpha) \sigma(\alpha, x) = 0$ for all $x \in X$.

*Proof.* If there exists a map $f\colon X \to \mathbb{R}$ with $\sum_x \sigma(\alpha, x) f(x) \leqslant \rho(\alpha)$ for all $\alpha \in A$, then for any $h\colon X \to \mathbb{R}$ of finite support and any nonnegative map $\tau\colon A \to \mathbb{R}$ of finite support with $\sum_\alpha \tau(\alpha) \sigma(\alpha, x) = h(x)$ for all $x \in X$ one necessarily has

$$\begin{aligned}\sum_\alpha \tau(\alpha) \rho(\alpha) &\geqslant \sum_\alpha \tau(\alpha) \sum_x \sigma(\alpha, x) f(x) \\ &= \sum_x \Big( \sum_\alpha \tau(\alpha) \sigma(\alpha, x) \Big) f(x) = \sum_x h(x) f(x) > -\infty.\end{aligned}$$

Vice versa, if $-\infty < i(h) = \inf(\sum_\alpha \tau(\alpha)\rho(\alpha) \mid \tau \in \mathbb{R}_h^A)$ for all $h: X \to \mathbb{R}$ of finite support, then the map $h \mapsto i(h)$ from the linear space $\mathbb{R}_0^X$ of all real valued maps, defined on $X$, of finite support is convex and positively homogeneous and so there exists a linear functional $l$: $\mathbb{R}_0^X \to \mathbb{R}$ with $l(h) \leqslant i(h)$ for all $h \in \mathbb{R}_0^X$. But any linear functional $l$: $\mathbb{R}_0^X \to \mathbb{R}$ is of the form $l = l_f$: $h \mapsto \sum_x h(x) f(x)$ for some map $f$: $X \to \mathbb{R}$ (namely the map $f$ defined by $f(x) = l(\delta_x)$ with $\delta_x$: $X \to \mathbb{R}$ defined by $\delta_x(y) = \delta_x^y$) and for this $f$ the inequality $l(h) \leqslant i(h)$ for all $h \in \mathbb{R}_0^X$ implies $\sum_{\alpha, x} \tau(\alpha)\sigma(\alpha, x) f(x) \leqslant \sum_\alpha \tau(\alpha)\rho(\alpha)$ for all nonnegative maps $\tau$: $A \to \mathbb{R}$ of finite support and thus $\sum_x \sigma(\alpha, x) f(x) \leqslant \rho(\alpha)$ for all $\alpha \in A$.

In particular, if $\mathbb{R}_h^A \neq \varnothing$ and thus $i(h) < +\infty$ for all $h \in \mathbb{R}_0^X$, then $i(0) = 0$ together with $i(h) + i(-h) \geqslant i(h + (-h)) = i(0) = 0$ implies $i(h) > -\infty$ for all $h \in \mathbb{R}_0^X$.

(5.2) *Remark.* Since any nonnegative map $\tau$: $A \to \mathbb{R}$ of finite support with $\sum_{\alpha \in A} \tau(\alpha)\sigma(\alpha, x) = 0$ for all $x \in X$ can obviously be written as a finite nonnegative linear combination of "minimal" such maps, i.e., such nonnegative maps $\tau$: $A \to \mathbb{R}$ of finite support with $\sum_{\alpha \in A} \tau(\alpha)\sigma(\alpha, x) = 0$ for all $x \in X$ for which $\mathrm{supp}(\tau') \subseteq \mathrm{supp}(\tau)$ for some nonnegative map $\tau'$: $A \to \mathbb{R}$ with $\sum_{\alpha \in A} \tau'(\alpha)\sigma(\alpha, x) = 0$ for all $x \in X$ implies $\mathrm{supp}(\tau') = \mathrm{supp}(\tau)$ (and thus $\tau' = c \cdot \tau$ for some $c > 0$!), it follows that there is some $f$: $X \to \mathbb{R}$ with $\sum_{x \in X} \sigma(\alpha, x) f(x) \leqslant \rho(\alpha)$ for all $\alpha \in A$ if $\mathbb{R}_h^A \neq \varnothing$ for all $h \in \mathbb{R}_0^X$ and $\sum_{\alpha \in A} \tau(\alpha)\rho(\alpha) \geqslant 0$ for all minimal nonnegative maps $\tau$: $A \to \mathbb{R}$ of finite support with $\sum_\alpha \tau(\alpha)\sigma(\alpha, x) = 0$ for all $x \in X$.

**(5.3)** Now assume $X$ to be a metric space. For any $f \in P_X$ and $\varepsilon \geqslant 0$ define $\mathscr{K}_f^\varepsilon =: \{(x, y) \in X \times X \mid f(x) + f(y) \leqslant xy + \varepsilon\}$ and $\mathscr{L}_f^\varepsilon = \{(x, y) \in X \times X \mid f(x) + f(y) < xy + \varepsilon\}$. Note that $\mathscr{K}_f^\varepsilon$ is a symmetric relation, defined on $X$, that $f \in T_X$ if and only if $\mathrm{supp}(\mathscr{K}_f^\varepsilon) = X$ for all $\varepsilon > 0$, that $\mathscr{L}_f^\varepsilon = \bigcup_{0 \leqslant \delta < \varepsilon} \mathscr{K}_f^\delta$ and that $\mathscr{K}_f = \mathscr{K}_f^0$.

The following statements are simple consequences of (5.1) and (5.2):

Let $\mathscr{K} \subseteq X \times X$ be a symmetric relation, defined on $X$ and let $\varepsilon, \eta \geqslant 0$ be two nonnegative real numbers. Then there exists some $f \in T_X$ with $\mathscr{K} \subseteq \mathscr{K}_f^\varepsilon$ if and only if for all $n \in \mathbb{N}$ and all $x_1, x_2, \dots, x_n$, $y_1, \dots, y_n = y_0 \in X$ with $\#\{x_1, \dots, x_n\} = \#\{y_1, \dots, y_n\} = n$ and $(x_1, y_1)$, $(x_2, y_2), \dots, (x_n, y_n) \in \mathscr{K}$ one has $\sum_{i=1}^n x_i y_{i-1} \leqslant \sum_{i=1}^n x_i y_i + n \cdot \varepsilon$ and there exists some $f \in P_X$ with $\mathscr{L}_f^\eta \subseteq \mathscr{K} \subseteq \mathscr{K}_f^\varepsilon$ if and only if for all $m, n \in \mathbb{N}$ and all $x_1, x_2, \dots, x_n$, $y_1$, $y_2, \dots, y_n = y_0 \in X$ with $\#\{x_1, \dots, x_n\} = \#\{y_1, \dots, y_n\} = n$, $(x_1, y_1)(x_2, y_2), \dots, (x_n, y_n) \in \mathscr{K}$, and $\#\{i \in \{1, \dots, n\} \mid (x_i, y_{i-1}) \notin \mathscr{K}\} = m$ one has $\sum_{i=1}^n x_i y_{i-1} + m \cdot \eta \leqslant \sum_{i=1}^n x_i y_i + n \cdot \varepsilon$.

Thus, if $\mathscr{K}$ is finite, there exists some $f \in P_X$ with $\mathscr{K} = \mathscr{K}_f$ if and only if for all $n \in \mathbb{N}$ and all $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_n = y_0 \in X$ with $\#\{x_1, \dots, x_n\} = \#\{y_1, \dots, y_n\} = n$ and $(x_1, y_1)$, $(x_2, y_2), \dots, (x_n, y_n) \in \mathscr{K}$ one has

$\sum_{i=1}^{n} x_i y_{i-1} \leqslant \sum_{i=1}^{n} x_i y_i$ with the inequality holding unless $(x_i, y_{i-1}) \in \mathscr{K}$ for all $i = 1, 2,..., n$.

In particular, if $n \in \mathbb{N}$, $x_i \in X$ for $i \in I =: \{\pm 1,..., \pm n\}$, and $\mathscr{K} = \{(x_i, x_{-i}) \mid i \in I\}$ then $\#\{x_1, x_{-1},..., x_n, x_{-n}\} = 2n$ and there exists some $f \in P_X$ with $\mathscr{K} = \mathscr{K}_f$ if and only if $\sum_{i \in I} x_i x_{-i} > \sum_{i \in I} x_i x_{\alpha(i)}$ for all permutations $\alpha$ of $I$ with $\alpha \neq -\mathrm{Id}_I$.

*Proof.* Let us observe at first that in view of (1.3) there exists some $f \in T_X$ with $\mathscr{K} \subseteq \mathscr{K}_f^{\varepsilon}$ if and only if there exists some $f \in P_X$ with $\mathscr{K} \subseteq \mathscr{K}_f^{\varepsilon}$.

Now let $A = A_{\mathscr{K}} =: \{(\{y_1, y_2\}, \vartheta) \in \mathscr{P}(X) \times \{\pm 1\} \mid \vartheta = -1$ or $(y_1, y_2) \in \mathscr{K}\}$, let $\sigma = \sigma_{\mathscr{K}}: A \times X \to \mathbb{R}$ be defined by $\sigma((\{y_1, y_2\}, \vartheta), x) = \vartheta(\delta_{y_1}^x + \delta_{y_2}^x)$ and define $\rho = \rho_{\mathscr{K}}^{\varepsilon}: A \to \mathbb{R}$ and $\rho' = \rho_{\mathscr{K}}^{\varepsilon,\eta}: A \to \mathbb{R}$ by $\rho((\{y_1, y_2\}, \vartheta)) = \vartheta \cdot y_1 y_2 + \varepsilon \cdot (\vartheta + 1)/2$ and $\rho'((\{y_1, y_2\}, \vartheta)) = \vartheta \cdot y_1 y_2 + \varepsilon \cdot ((\vartheta + 1)/2) + \eta \cdot ((\vartheta - 1)/2) \cdot \chi_{\mathscr{K}}(y_1, y_2)$ with

$$\begin{aligned} \chi_{\mathscr{K}}(y_1, y_2) &= 1 \quad \text{for} \quad (y_1, y_2) \notin \mathscr{K}, \\ &= 0 \quad \text{for} \quad (y_1, y_2) \in \mathscr{K}. \end{aligned}$$

Then we have $\mathscr{K} \subseteq \mathscr{K}_f^{\varepsilon}$ for some $f \in P_X$ if and only if $\sum_{x \in X} \sigma((\{y_1, y_2\}, \vartheta), x) \cdot f(x) \leqslant \rho((\{y_1, y_2\}, \vartheta))$ for all $(\{y_1, y_2\}, \vartheta) \in A$ and we have $f \in P_X$ and $\mathscr{L}_f^{\eta} \subseteq \mathscr{K} \subseteq \mathscr{K}_f^{\varepsilon}$ for some $f \in \mathbb{R}^X$ if and only if $\sum_{x \in X} \sigma((\{y_1, y_2\}, \vartheta), x) \cdot f(x) \leqslant \rho'((\{y_1, y_2\}, \vartheta))$ for all $(\{y_1, y_2\}, \vartheta) \in A$.

Now consider the set $H_\sigma = \{h \in \mathbb{R}_0^X \mid \mathbb{R}_h^A \neq \varnothing\}$. Obviously, $H_\sigma$ is the convex cone in $\mathbb{R}_0^X$ which is spanned by the maps $\sigma_\alpha: X \to \mathbb{R}: x \mapsto \sigma(\alpha, x)$ $(\alpha \in A)$ and so it contains $-\delta_y = \frac{1}{2}\sigma_{(\{y,y\},-1)}$ for all $y \in X$ as well as $\delta_x = \sigma_{(\{y,x\}+1)} + \frac{1}{2}\sigma_{(\{y,y\},-1)}$ for all $x \in \operatorname{supp}(\mathscr{K})$. Thus, if we assume for a moment that $X = \operatorname{supp}(\mathscr{K})$ then it follows immediately that $\mathbb{R}_h^A \neq \varnothing$ for all $h \in \mathbb{R}_0^X$ and so each of our two systems of inequalities can be solved by some $f \in \mathbb{R}^X$ if and only if for any minimal nonnegative map $\tau: A \to \mathbb{R}$ of finite support with $\sum_{\alpha \in A} \tau(\alpha)\, \sigma(\alpha, x) = 0$ for all $x \in X$ one has $\sum_{\alpha \in A} \tau(\alpha)\, \rho(\alpha) \geqslant 0$ or $\sum_{\alpha \in A} \tau(\alpha)\, \rho'(\alpha) \geqslant 0$, respectively.

Thus for the proof of the first two statements it is enough to show that for any such map $\tau$ there exist elements $n \in \mathbb{N}$, $x_1, x_2,..., x_n$, $y_1$, $y_2,..., y_n = y_0 \in X$ and $c > 0$ with $n = \#\{x_1, x_2,..., x_n\} = \#\{y_1,..., y_n\}$, $(x_1, y_1)$, $(x_2, y_2),..., (x_n, y_n) \in \mathscr{K}$,

$$\tau((\{z_1, z_2\}, \vartheta)) = c \ \text{ if } \ (\{z_1, z_2\}, \vartheta) \in \{(\{x_i, y_i\}, +1) \mid i = 1,..., n\} \cup \{(\{x_i, y_{i-1}\}, -1) \mid i = 1,..., n\}$$

and $\tau((\{z_1, z_2\}, \vartheta)) = 0$ otherwise. But for any nonnegative map $\tau: A \to \mathbb{R}$ of finite support with $\sum_{\alpha \in A} \tau(\alpha)\, \sigma(\alpha, x) = 0$ for all $x \in X$ one has

$$\begin{aligned} &\{x \in X \mid (x, y) \in \mathscr{K} \text{ and } \tau((\{x, y\}, +1)) > 0 \text{ for some } y \in X\} \\ &\quad = \{x \in X \mid \tau((\{x, y\}, -1)) > 0 \text{ for some } y \in X\}. \end{aligned}$$

Thus, if $X_\tau$ denotes this subset of $X$, there are maps $a\colon X_\tau \to X_\tau$ and $b\colon X_\tau \to X_\tau$ with $(x, a(x)) \in \mathscr{K}$, $\tau((\{x, a(x)\}, +1)) > 0$, and $\tau((\{x, b(x)\}, -1)) > 0$ for all $x \in X$. Since $X_\tau$ is finite, there is some $x \in X_\tau$ and some $n \in \mathbb{N}$ with $x = (ba)^n(x) \neq (ba)^i(x)$ for all $i \in \{1,..., n-1\}$ unless $X_\tau = \varnothing$ and $\tau \equiv 0$. Thus with $x_i = (ba)^{i-1}(x)$ and $y_i = a(x_i)$—and, hence, $x_i = b(y_{i-1})$—for $i = 1,..., n$ one may define

$$\begin{aligned} \tau' : A \to \mathbb{R} : (\{z_1, z_2\}, \vartheta) &\mapsto +1 \quad \text{if } (\{z_1, z_2\}, \vartheta) \in \{(\{x_i, y_i\}, +1) \mid i = 1,..., n\} \\ &\qquad\qquad \cup \{(\{x_i, y_{i-1}\}, -1) \mid i = 1,..., n\} \\ &\mapsto 0 \quad \text{otherwise} \end{aligned}$$

One checks easily that $\tau'$ is a nonnegative map of finite support with $\sum_\alpha \tau'(\alpha)\,\sigma(\alpha, x) = 0$ for all $x \in X$ and that $\operatorname{supp}(\tau') \subseteq \operatorname{supp}(\tau)$. Thus we have indeed $\tau = c \cdot \tau'$ for some $c > 0$ in case $\tau$ is supposed to be minimal.

If we do not suppose $X = \operatorname{supp}(\mathscr{K})$ the above argument shows that at least there is a map $f' : Y = \operatorname{supp}(\mathscr{K}) \to \mathbb{R}$ with $f'(x) + f'(y) \geqslant xy$ for all $x, y \in Y$ and $f'(x) + f'(y) \leqslant xy + \varepsilon$ for all $x, y \in X$ with $(x, y) \in \mathscr{K}$ (and $f'(x) + f'(y) \geqslant xy + \eta$ for all $x, y \in Y$ with $(x, y) \notin \mathscr{K}$, respectively), so we may extend this map to a solution $f\colon X \to \mathbb{R}$ of our original problem by choosing some fixed $y_0 \in Y$ and defining

$$\begin{aligned} f(x) &= f'(x) && \text{if } x \in Y, \\ &= xy_0 + f'(y_0) + \eta && \text{if } x \in X \backslash Y. \end{aligned}$$

Now assume $\mathscr{K}$ to be finite. As above we may assume w.l.o.g. that $X = \operatorname{supp}(\mathscr{K})$. There exists some $f \in P_X$ with $\mathscr{K} = \mathscr{K}_f$ if and only if there is some $f \in P_X$ and some $\eta > 0$ with $\mathscr{L}_f^\eta \subseteq \mathscr{K} \subseteq \mathscr{K}_f$ and, thus, there is some $f \in P_X$ with $\mathscr{K} = \mathscr{K}_f$ if and only if there is some $\eta > 0$ with $\sum_{i=1}^n x_i y_{i-1} \leqslant \sum_{i=1}^n x_i y_i - m \cdot \eta$ for all $m, n \in \mathbb{N}$ and $x_1, x_2,..., x_n$, $y_1,..., y_n = y_0 \in X$ with $\#\{x_1,..., x_n\} = \#\{y_1,..., y_n\} = n$, $(x_1, y_1),...,$ $(x_n, y_n) \in \mathscr{K}$ and $\#\{i \in \{1,..., n\} \mid (x_i, y_{i-1}) \notin \mathscr{K}\} = m$ which in view of the finiteness of $X$ is obviously equivalent with

$$\sum_{i=1}^n x_i y_{i-1} \leqslant \sum_{i=1}^n x_i y_i \qquad \text{for all such } x_1,..., x_n, y_1,..., y_n = y_0 \in X,$$

$$\sum_{i=1}^n x_i y_{i-1} < \sum_{i=1}^n x_i y_i \qquad \text{unless } (x_i, y_{i-1}) \in \mathscr{K} \text{ for all } i = 1,..., n.$$

Now assume $n \in \mathbb{N}$, $x_i \in X$ for $i \in I$, and $\mathscr{K} = \{(x_i, x_{-i}) \mid i \in I\}$. If

$\#\{x_1,..., x_n, x_{-1},..., x_{-n}\} = 2n$ and there exists some $f \in P_X$ with $\mathscr{K}_f = \mathscr{K}$ we obviously have

$$\sum_{i\in I} x_i x_{-i} = \sum_{i\in I} (f(x_i) + f(x_{-i})) = \sum_{i\in I} (f(x_i) + f(x_{\alpha(i)})) > \sum_{i\in I} x_i x_{\alpha(i)}$$

for all permutations $\alpha$ of $I$ with $\alpha \neq -\mathrm{Id}_I$.

Now assume this condition to be fulfilled. Since this implies readily $\#\{x_1, x_{-1},..., x_n, x_{-n}\} = 2n$, it is obviously enough to show that for all $m \in \mathbb{N}$ and $z_1,..., z_m, y_1,..., y_m = y_0 \in X$ with $=\{z_1,..., z_m\} = \#\{y_1,..., y_m\} = m$ and $(z_i, y_i) \in \mathscr{K}$ for all $i = 1,..., m$ one has $\sum_{i=1}^n z_i y_i > \sum_{i=1}^m z_i y_{i-1}$ unless $m = 1$. So assume $m > 1$ and let $\beta$: $\{1,..., m\} \to I$ be the unique map with $z_i = x_{\beta(i)}$ for all $i \in \{1,..., m\}$ so $\beta$ is necessarily injective since $\#\{z_1,..., z_m\} = m$ and we have $y_i = x_{-\beta(i)}$ for all $i \in \{1,..., m\}$. Define $\alpha: I \simeq I$ by

$$\begin{aligned} \alpha(j) &= -\beta(i-1) && \text{if } j = \beta(i) \text{ for some } i \in \{2,..., m\}, \\ &= -\beta(m) && \text{if } j = \beta(1), \\ &= -j && \text{if } j \notin \{\beta(1),..., \beta(m)\}; \end{aligned}$$

$\alpha$ is obviously bijective and one has $\alpha \neq -\mathrm{Id}_I$ since $m > 1$ and so, for instance, $\alpha(\beta(1)) = -\beta(m) \neq -\beta(1)$. Thus we get $\sum_{j\in\{\pm 1,...,\pm n\}} x_j x_{-j} > \sum_{j\in\{\pm 1,...,\pm n\}} x_j x_{\alpha(j)}$ which implies

$$\sum_{j\in\beta(\{1,...,m\})} x_j x_{-j} > \sum_{j\in\beta(\{1,...,m\})} x_j x_{\alpha(j)},$$

i.e.,

$$\sum_{i=1}^{m} z_i y_i > \sum_{i=1}^{m} z_i y_{i-1}. \qquad \text{Q.E.D.}$$

(5.4) *Remark.* Note that we used the triangular inequality only to extend a solution $f'$: $Y = \mathrm{supp}\,\mathscr{K} \to \mathbb{R}$ to a solution $f$: $X \to \mathbb{R}$. More precisely, it follows from the above proof of (5.3) that its statement are correct for any symmetric map $X \times X \to \mathbb{R}$: $(x, y) \mapsto xy$ with $s_{x,y} =: \sup(xz - yz \mid z \in X) < +\infty$ for all $x, y \in X$, since under such conditions one can always extend a solution $f'$: $Y = \mathrm{supp}\,\mathscr{K} \to \mathbb{R}$ to a solution $f$: $X \to \mathbb{R}$ after choosing some fixed $y_0 \in Y$ by

$$\begin{aligned} f(x) &= f'(x) && \text{for } x \in Y, \\ &= \eta + f'(y_0) + s_{x,y_0} && \text{for } x \in \backslash Y, \end{aligned}$$

since for this $f$ one has for $x \in X \backslash Y$ and $z \in Y$ the relation $f(x) + f(z) = \eta + f'(y_0) + s_{x,y_0} + f'(z) \geqslant \eta + (f'(y_0) + f'(z)) + s_{x,y_0} \geqslant \eta + y_0 z + (xz - y_0 z) \geqslant xz + \eta$ and for $x, z \in X \backslash Y$ one has $f(x) + f(z) =$

$2\eta + 2f'(y_0) + s_{x,y_0} + s_{z,y_0} \geqslant \eta + 2f'(y_0) + (xz - y_0 z) + (zy_0 - y_0 y_0) = \eta + xz + (f'(y_0) + f'(y_0) - y_0 y_0) \geqslant \eta + xz.$

Note that $s_{x,y} = \sup(xz - yz \mid z \in X) = xy$ for all $x, y \in X$ if and only if the map $X \times X \to \mathbb{R}$: $(x, y) \mapsto xy$ is essentially a metric, i.e., it satisfies all the conditions defining a metric except possibly the condition $xy = 0 \Rightarrow x = y$, and thus one has $s_{x,y} < +\infty$ for all $x, y \in X$ whenever the map $(x, y) \mapsto xy$ is the sum of two maps $(x, y) \mapsto xy_1$ and $(x, y) \mapsto xy_2$, one of which is a metric whereas the other one is bounded.

The example $X = \{a; b_1, b_2, \dots, b_n, \dots\}$,

$$\begin{aligned} xy &= 0 && \text{if } x = y, \\ &= 2 && \text{if } x \neq y \text{ and } x, y \in \{b_1, b_2, \dots\}, \\ &= n && \text{if } x = a \text{ and } y = b_n, \end{aligned}$$

and $\mathscr{K} = \{(b_n, b_m) \mid n \neq m\}$ shows that without some extra conditions concerning the map $(x, y) \mapsto xy$ there may be no possibility to extend a solution $f'$: $Y = \operatorname{supp} \mathscr{K} \to \mathbb{R}$ to a solution $f$: $X \to \mathbb{R}$: there is only one solution $f'$: $Y = \operatorname{supp} \mathscr{K} = \{b_1, b_2, \dots\} \to \mathbb{R}$ with $f'(b_n) + f'(b_m) = 2$ for all $(b_n, b_m) \in \mathscr{K}$, namely, $f'(b_n) = 1$ for all $n = 1, 2, \dots$, whereas the inequalities $f(a) + f'(b_n) = f(a) + 1 \geqslant n$ for all $n = 1, 2, \dots$, do not admit a solution $f(a) \in \mathbb{R}$.

**(5.5)** Another consequence of these results can be stated once we define for any symmetric relation $\mathscr{K} \subseteq X \times X$ and $x, y \in X$ the expression

$$xy_{\mathscr{K}} =: \inf\Big(\sum_{i=1}^{n} x_i y_i - \sum_{i=2}^{n} x_i y_{i-1} \mid n \in \mathbb{N}; x = x_1, \dots, x_n, y_1, \dots, y_n = y \in X;$$
$$\#\{x_1, \dots, x_n\} = \#\{y_1, \dots, y_n\} = n; (x_1, y_1), \dots, (x_n, y_n) \in \mathscr{K}\Big).$$

Note that $xy_{\mathscr{K}} \leqslant xy$ for $(x, y) \in \mathscr{K}$ since for $n = 1$, $x_1 = x$, $y_1 = y$, one has $xy = \sum_{i=1}^{n} x_i y_i - \sum_{i=2}^{n} x_i y_{i-1}$. If $\mathscr{K} \subseteq \mathscr{K}_f$ for some $f \in P_X$, then one has $xy_{\mathscr{K}} \geqslant f(x) + f(y)$ for all $x, y \in X$ since $(x_1, y_1), \dots, (x_n, y_n) \in \mathscr{K}$ implies $\sum_{i=1}^{n} x_i y_i - \sum_{i=2}^{n} x_i y_{i-1} \geqslant \sum_{i=1}^{n} (f(x_i) + f(y_i)) - \sum_{i=2}^{n} (f(x_i) + f(y_{i-1})) = f(x_1) + f(y_n)$.

It follows directly from (5.3) that $\mathscr{K} \subseteq \mathscr{K}_f$ for some $f \in P_X$ if and only if $xy_{\mathscr{K}} \geqslant xy$ for all $x, y \in X$. Moreover, in this case we have

$$\bigcap_{f \in P_X, \mathscr{K} \subseteq \mathscr{K}_f} \mathscr{K}_f = \{(x, y) \in X^2 \mid xy_{\mathscr{K}} = xy\},$$

since $xy_{\mathscr{K}} = xy$ and $\mathscr{K} \subseteq \mathscr{K}_f$ implies $xy \leqslant f(x) + f(y) \leqslant xy_{\mathscr{K}} = xy$ and thus

$xy = f(x) + f(y)$, i.e., $(x, y) \in \mathscr{K}_f$, whereas $xy_{\mathscr{K}} > xy$ for some $x, y \in X$ implies $(x, y) \notin \mathscr{K}$ and so, using (5.4) with respect to the map

$$X \times X \to \mathbb{R} : (a, b) \mapsto ab \qquad \text{if} \quad \{a, b\} \neq \{x, y\},$$
$$\mapsto \operatorname{Min}(xy_{\mathscr{K}}, xy + 1) \qquad \text{if} \quad \{a, b\} = \{x, y\},$$

one finds easily some $f \in P_X$ with $\mathscr{K} \subseteq \mathscr{K}_f$ and $f(x) + f(y) \geqslant \operatorname{Min}(xy_{\mathscr{K}}, xy + 1) > xy$, i.e., $(x, y) \notin \mathscr{K}_f$.

*Remark.* Since $f_1, f_2, \ldots, f_n, \ldots, \in P_X$ implies $f = \sum_{n=1}^{\infty} (1/2^n) f_n \in P_X$ and $\mathscr{K}_f = \bigcap_{n=1}^{\infty} \mathscr{K}_{f_n}$ it follows from (5.5) that for a symmetric relation $\mathscr{K} \subseteq X \times X$ with countable support $\operatorname{supp} \mathscr{K} \subseteq X$ there exists some $f \in P_X$ with $\mathscr{K} = \mathscr{K}_f$ if and only if $\sum_{i=1}^{n} x_i y_i \geqslant \sum_{i=1}^{n} x_i y_{i-1}$ for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n = y_0 \in X$ with $\#\{x_1, \ldots, x_n\} = \#\{y_1, \ldots, y_n\} = n$, $(x_1, y_1), \ldots, (x_n, y_n) \in \mathscr{K}$, and $\mathscr{K} \supseteq \{(x, y) \in X \times X \mid xy_{\mathscr{K}} = xy\}$.

I do not know whether the same holds without the countability condition, but I doubt it.

**(5.6)** As above, let $\mathscr{K} \subseteq X \times X$ denote a symmetric relation defined on $X$. If $\mathscr{L} \subseteq X \times X$ is another relation defined on $X$, we define a sequence $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in X^{2n}$ to be $\mathscr{K}$-$\mathscr{L}$-sequence if $(x_1, y_1)$, $(x_2, x_2), \ldots, (x_n, y_n) \in \mathscr{L}$ and $(y_1, x_2)$, $(y_2, x_3), \ldots, (y_{n-1}, x_n)$, $(y_n, x_1) \in \mathscr{K}$. Then the following three statements are equivalent:

(i) For any $\mathscr{K}$-$\mathscr{L}$-sequence $(x_1, y_1, \ldots, x_n, y_n)$ with $\#\{x_1, \ldots, x_n\} = \#\{y_1, \ldots, y_n\} = n$ we have $(y_n, x_1) \in \mathscr{L}$.

(ii) For any $\mathscr{K}$-$\mathscr{L}$-sequence $(x_1, y_1, \ldots, x_n, y_n)$ we have $(y_n, x_1) \in \mathscr{L}$.

(iii) For any $\mathscr{K}$-$\mathscr{L}$-sequence $(x_1, y_1, \ldots, x_n, y_n)$ we have $(y_1, x_2), \ldots, (y_{n-1}, x_n)$, $(y_n, x_1) \in \mathscr{L}$.

*Proof.* Since (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious, we only have to show that (i) implies (iii). If not, let $(x_1, y_1, \ldots, x_n, y_n)$ be a $\mathscr{K}$-$\mathscr{L}$-sequence with $\{(y_1, x_2), \ldots, (y_n, x_1)\} \nsubseteq \mathscr{L}$ of smallest length.

Since $(x_i, y_i, \ldots, x_n, y_n, x_1, y_1, \ldots, x_{i-1}, y_{i-1})$ is a $\mathscr{K}$-$\mathscr{L}$-sequence for each $i \in \{1, \ldots, n\}$, we may assume w.l.o.g. that $(y_n, x_1) \notin \mathscr{L}$ and so we necessarily have $\#\{x_1, \ldots, x_n\} < n$ or $\#\{y_1, \ldots, y_n\} < n$. But if $x_i = x_j$ or $y_i = y_j$ for some $i, j \in \{1, \ldots, n\}$ with $1 \leqslant i < j \leqslant n$, then $(x_1, y_1, \ldots, x_{i-1}, y_{i-1}, x_i = x_j, y_j, x_{j+1}, y_{j+1}, \ldots, x_n, y_n)$ or $(x_1, y_1, \ldots, x_i, y_i = y_j, x_{j+1}, y_{j+1}, \ldots, x_n, y_n)$ are $\mathscr{K}$-$\mathscr{L}$-sequence of length $n - j + i < n$, respectively, and so we have $(y_n, x_1) \in \mathscr{L}$, a contradiction.

We define a relation $\mathscr{L} \subseteq X \times X$ to be $\mathscr{K}$-closed, if it is symmetric and satisfies the three equivalent conditions, stated above.

Thus a symmetric relation $\mathscr{L} \subseteq X \times X$ is $\mathscr{K}$-closed, if and only if

$\mathscr{K} \cap \mathscr{L}(\mathscr{K}\mathscr{L})^n \subseteq \mathscr{L}$ for any $n \in \mathbb{N}$, where, as above, for two relations $\mathscr{A}, \mathscr{B} \subseteq X \times X$ we denote by $\mathscr{A}\mathscr{B}$ the relational product $\mathscr{A}\mathscr{B} =: \{(x, y) \in X \times X \mid \text{there is some } z \in X \text{ with } (x, z) \in \mathscr{A} \text{ and } (z, y) \in \mathscr{B}\}$, or—still in other words—it is $\mathscr{K}$-closed if and only if for all $n, m \neq \mathbb{N}$ with $n \equiv m(2)$ one has

$$\underbrace{\mathscr{K}\mathscr{L}\mathscr{K}\mathscr{L}\mathscr{K}\ldots}_{n\text{-factors}} \cap \underbrace{\mathscr{L}\mathscr{K}\mathscr{L}\mathscr{K}\mathscr{L}\ldots}_{m\text{-factors}} \subseteq \mathscr{L}^n.$$

**(5.7)** The following observations are more or less obvious:

(a) If $\mathscr{K} \subseteq \mathscr{L} \subseteq X \times X$ and if $\mathscr{L}$ is symmetric, then $\mathscr{L}$ is $\mathscr{K}$-closed.

(b) If $\mathscr{K}_1 \subseteq \mathscr{K}_2 \subseteq X \times X$ are two symmetric relations and if $\mathscr{L} \subseteq X \times X$ is $\mathscr{K}_2$-closed, then it is $\mathscr{K}_1$-closed.

(c) If $\mathscr{L} \subseteq X \times X$ is $\mathscr{K}$-closed and if $\mathscr{L}' \subseteq \mathscr{L}$ is $\mathscr{L}$-closed, then $\mathscr{L}'$ is $\mathscr{K}$-closed, too.

(d) The intersection of $\mathscr{K}$-closed relations is $\mathscr{K}$-closed, in particular, for any relation $\mathscr{L} \subseteq X \times X$ there exists a $\mathscr{K}$-closure $\bar{\mathscr{L}} = \bar{\mathscr{L}}_{\mathscr{K}} = \bigcap_{\mathscr{L}' \mathscr{K}\text{-closed}, \mathscr{L} \subseteq \mathscr{L}'} \mathscr{L}'$.

(e) For any $Y \subseteq X$ and any $\mathscr{K}$-closed relation $\mathscr{L} \subseteq X \times X$ the relation $\mathscr{L} \cap (Y \times Y)$ is $\mathscr{K}$-closed, too, in particular, one has $\operatorname{supp}(\mathscr{L}) = \operatorname{supp}(\bar{\mathscr{L}})$ for any $\mathscr{L} \subseteq X \times X$.

(f) If $\mathscr{L} \subseteq \mathscr{K}$ is $\mathscr{K}$-closed, then we have $\mathscr{L}^{2n+1} \subseteq \mathscr{L}(\mathscr{K}\mathscr{L})^n$ and $\mathscr{L}^{2n+1}\mathscr{K}\mathscr{L}^{2m+1} \subseteq \mathscr{L}(\mathscr{K}\mathscr{L})^{n+m+1}$ and thus $\mathscr{L}^{2n+1} \cap \mathscr{K} \subseteq \mathscr{L}$ and $\mathscr{L}^{2n+1}\mathscr{K}\mathscr{L}^{2m+1} \cap \mathscr{K} \subseteq \mathscr{L}$. In particular, if $(x, y) \in \mathscr{K}\backslash\mathscr{L}$ then $(x, y) \in \mathscr{L}^k$ implies $k \equiv 0(2)$ and $(x, x) \in \mathscr{L}^k$, $(y, y) \in \mathscr{L}^j$ implies $k \cdot j \equiv 0(2)$, or, in other words, if $(x, y) \in \mathscr{K}\backslash\mathscr{L}$ and if $x$ and $y$ are in the same connected component of $\mathscr{L}$, then this component is necessarily bipartite and any path from $x$ to $y$ in $\mathscr{L}$ has even length. Whereas if $x$ and $y$ are in different connected components of $\mathscr{L}$, then at least one of these two components is bipartite, or—still in other words—if $\mathscr{L} \subsetneq \mathscr{K}$ is $\mathscr{K}$-closed and if $\operatorname{supp} \mathscr{L} = \operatorname{supp} \mathscr{K}$, then the vectorspace $W_{\mathscr{K}} =: \{v \in \mathbb{R}^{\operatorname{supp} \mathscr{K}} \mid v(x) + v(y) = 0 \text{ for all } (x, y) \in \mathscr{K}\}$, whose dimension measures the number of bipartite connected components of $\mathscr{K}$, is properly contained in the correspondingly defined vectorspace $W_{\mathscr{L}} =: \{v \in \mathbb{R}^{\operatorname{supp} \mathscr{L}} \mid v(x) + v(y) = 0 \text{ for all } (x, y) \in \mathscr{L}\}$.

**(5.8)** If $v: X \to \mathbb{R}$ satisfies $v(x) + v(y) \geqslant 0$ for all $(x, y) \in \mathscr{K}$, then any symmetric relation $\mathscr{L} \subseteq X \times X$ with $v(x) + v(y) \leqslant 0$ for all $(x, y) \in \mathscr{L}$ and $\mathscr{L} \supseteq \mathscr{L}_v^{\mathscr{K}} =: \{(x, y) \in \mathscr{K} \mid v(x) + v(y) = 0\}$ is $\mathscr{K}$-closed, since $x_1, \ldots, x_n$, $y_1, \ldots, y_n = y_0 \in X$, $v(x_i) + v(y_i) \leqslant 0$, and $v(x_i) + v(y_{i-1}) \geqslant 0$ for all $i = 1, \ldots, n$, implies $v(x_i) + v(y_{i-1}) = 0$ for all $i = 1, \ldots, n$. So, in particular, $\mathscr{L}_v^{\mathscr{K}}$ is $\mathscr{K}$-closed for any such $v \in \mathbb{R}^X$.

Vice versa, if $\mathscr{L} \subseteq \mathscr{K}$ is $\mathscr{K}$-closed, then $\mathscr{L}$ is the intersection of all $\mathscr{L}_v^{\mathscr{K}}$, where $v$ runs through the set $W_{\mathscr{L}}^{\mathscr{K}}$ consisting of all maps in $\mathbb{R}^X$ with $v(x)+v(y) \geqslant 0$ for all $(x, y) \in \mathscr{K}$ and $v(x)+v(y)=0$ for all $(x, y) \in \mathscr{L}$. This could be derived from (5.1) and (5.2), but a direct proof is more instructive. Define a pair $(Y, v)$ with $Y \subseteq X$ and $v \in \mathbb{R}^Y$ to be admissible if $Y$ contains all $x \in X$ with $(x, x) \in \mathscr{L}^{(1)} =: \bigcup_{n \in \mathbb{N}} \mathscr{L}^{2n+1}$, if $(y, z) \in (Y \times Y) \cap \mathscr{L}^n$ implies $v(y)=(-1)^n v(z)$ and if $(y, z) \in (Y \times Y) \cap \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$ implies $(-1)^m v(y)+(-1)^j v(z) \geqslant 0$ $(n, m, k, j \in \mathbb{N})$. Note that $(Y, w|_Y)$ is admissible for any $w \in W_{\mathscr{L}}^{\mathscr{K}}$.

It follows that for $x \in X \backslash Y$ and $w \in \mathbb{R}^{Y \cup \{x\}}$ the pair $(Y \cup \{x\}, w)$ is admissible if and only if $(Y, w|_Y)$ is admissible and $w(x)$ satisfies the following conditions: $w(x)=(-1)^n w(y)$ for all $y \in Y$ and $n \in \mathbb{N}$ with $(x, y) \in \mathscr{L}^n$, $(-1)^m w(x)+(-1)^j w(z) \geqslant 0$ for all $z \in Y$, and $m, j \in \mathbb{N}$ with $(x, z) \in \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$ for some $k \in \mathbb{N}$ and $(-1)^m w(x) \geqslant 0$ whenever $(x, x) \in \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$ for some $k, j \in \mathbb{N}$ with $j \equiv m(2)$.

But for any admissible pair $(Y, v)$ and any $x \in X \backslash Y$ one has the following implications:

(i) $y, z \in Y$; $n, m \in \mathbb{N}$; $(x, y) \in \mathscr{L}^n$; and $(x, z) \in \mathscr{L}^m$ together imply $(y, z) \in \mathscr{L}^{n+m}$ and thus $(-1)^n v(y)=(-1)^m v(z)$;

(ii) $y, z \in Y$; $n, m, k, j \in \mathbb{N}$; $(x, y) \in \mathscr{L}^n$; and $(x, z) \in \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$ together imply $(y, z) \in \mathscr{L}^{n+m} \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$ and thus $(-1)^m((-1)^n v(y))+(-1)^j v(z) \geqslant 0$;

(iii) $y \in Y$; $n, m, k, j \in \mathbb{N}$; $k \equiv j(2)$; $(x, y) \in \mathscr{L}^n$; and $(x, x) \in \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \cdot \mathscr{L}^j$ together imply $(y, y) \in \mathscr{L}^{m+n} \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^{j+n}$ and thus $(-1)^m((-1)^n v(y)) \geqslant 0$;

(iv) $y, z \in Y$; $m, k, j, a, b, c \in \mathbb{N}$; $(x, y) \in \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$; $(x, z) \in \mathscr{L}^a \mathscr{K}(\mathscr{L}\mathscr{K})^b \mathscr{L}^c$; and $m+a=2n+1$ for some $n \in \mathbb{N}$ together imply $(y, z) \in \mathscr{L}^j \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^{m+a} \mathscr{K}(\mathscr{L}\mathscr{K})^b \mathscr{L}^c \subseteq \mathscr{L}^j \mathscr{K}(\mathscr{L}\mathscr{K})^{k+n+1+b} \cdot \mathscr{L}^c$ and thus $(-1)^j v(y)+(-1)^c v(z) \geqslant 0$, so we have $s_1(x, Y) =: \sup((-1)^{j+1} v(y) \mid y \in Y,\ j \in \mathbb{N}$, there exists $m, k \in \mathbb{N}$ with $(x, y) \in \mathscr{L}^{2m} \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j) \leqslant s_2(x, Y) =: \inf((-1)^c v(z) \mid z \in Y,\ c \in \mathbb{N}$, there exists $a, b \in \mathbb{N}$ with $(x, z) \in \mathscr{L}^{2a+1} \mathscr{K}(\mathscr{L}\mathscr{K})^b \mathscr{L}^c)$.

(v) $y \in Y$; $m, k, j, a, b, c \in \mathbb{N}$; $a \equiv c(2)$; $m \not\equiv a(2)$; $(x, y) \in \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$; and $(x, x) \in \mathscr{L}^a \mathscr{K}(\mathscr{L}\mathscr{K})^b \mathscr{L}^c$ together imply $(y, y) \in \mathscr{L}^j \mathscr{K}(\mathscr{L}\mathscr{K})^n \mathscr{L}^j$ (with $n=k+((m+a+1)/2)+b+((c+m+1)/2)+k \in \mathbb{N}$) and thus $(-1)^j v(y) \geqslant 0$, so in case $a \equiv 1(2)$ we have $s_1(x, Y) \leqslant 0$ and in case $a \equiv 0(2)$ we have $s_2(x, Y) \geqslant 0$.

(vi) $m, k, j, a, b, c \in \mathbb{N}$; $m \equiv j(2)$; $a \equiv c(2)$; $(x, x) \in \mathscr{L}^m \mathscr{K}(\mathscr{L}\mathscr{K})^k \mathscr{L}^j$; and $(x, x) \in \mathscr{L}^a \mathscr{K}(\mathscr{L}\mathscr{K})^b \mathscr{L}^c$ implies $m \equiv a(2)$, since otherwise we may assume $m=2n$ and $a=2d+1$ $(n, d \in \mathbb{N})$ from which we get $(x,x) \in (\mathscr{K}\mathscr{L})^{n+k+(j/2)}\mathscr{K}$ as well as $(x,x) \in \mathscr{L}(\mathscr{K}\mathscr{L})^{d+b+((c+1)/2)}$ which implies $(x,x) \in \mathscr{L}^{(1)}$ for a $\mathscr{K}$-closed relation $\mathscr{L} \subseteq \mathscr{K}$.

It now follows easily that any admissible pair $(Y, v)$ with $|v(y)| \leqslant 1$ for all $y \in Y$ can be extended to an admissible pair $(Y \cup \{x\}, w)$ with $w|_Y = v$ and $|w(x)| \leqslant 1$ for any $x \in X$ by putting $w(x) = (-1)^n v(y)$ if there exists some $y \in Y$ and $n \in \mathbb{N}$ with $(x, y) \in \mathscr{L}^n$ and by choosing $w(x)$ arbitrarily in $[-1, +1] \cap [s_1(x, Y), s_2(x, Y)]$ if no such $y$ exists and no $m, k, j \in \mathbb{N}$ with $(x,x) \in \mathscr{L}^m\mathscr{K}(\mathscr{L}\mathscr{K})^k\mathscr{L}^j$, whereas in the latter case we have to choose $w(x)$ in $[-1, 0] \cap [s_1(x, Y), s_2(x, Y)]$ if $m \equiv 1(2)$ and in $[0, +1] \cap [s_1(x, Y), s_2(x, Y)]$ if $m \equiv O(2)$. Moreover, if $v(Y) \subseteq \{0, \pm 1\}$, we may also choose $w(x) \in \{0, \pm 1\}$.

Thus it follows from Zorn's lemma, that a pair $(Y, v)$ with $v(Y) \subseteq [-1, +1]$ (or with $v(Y) \subseteq \{0, \pm 1\}$) is admissible if and only if $v = w|_Y$ for some $w \in W_{\mathscr{L}}^{\mathscr{K}}$ with $v(X) \subseteq [-1, +1]$ (or with $v(X) \subseteq \{0, \pm 1\}$, respectively) and $Y \supseteq \{x \in X \mid (x, x) \in \mathscr{L}^{(1)}\}$.

Now assume $\mathscr{L} \subseteq \mathscr{K}$ to be $\mathscr{K}$-closed and $(x_0, y_0) \in \mathscr{K} \backslash \mathscr{L}$. We claim the existence of $w \in W_{\mathscr{L}}^{\mathscr{K}}$ with $w(X) \subseteq \{0, \pm 1\}$ and with $w(x_0) + w(y_0) > 0$. In view of the above results it is enough to show the existence of an admissible pair $(Y, v)$ with $x_0, y_0 \in Y$, $v(Y) \subseteq \{0, \pm 1\}$, and $v(x_0) + v(y_0) > 0$. So put $Y = \{x_0, y_0\} \cup \{x \in X \mid (x, x) \in \mathscr{L}^{(1)}\}$, put $v(y) = 0$ if $y \in Y$ and $(y, y) \in (\mathscr{L}\mathscr{K})^k\mathscr{L}$ for some $k \in \mathbb{N}$, in particular, if $(y, y) \in \mathscr{L}^{(1)}$, otherwise put $v(y) = 1$. Then we have $v(x_0) + v(y_0) > 0$ since $(x_0, x_0) \in (\mathscr{L}\mathscr{K})^k\mathscr{L}$ and $(y_0, y_0) \in (\mathscr{L}\mathscr{K})^j\mathscr{L}$ implies $(x_0, y_0) \in \mathscr{K} \cap (\mathscr{L}\mathscr{K})^k\mathscr{L}\mathscr{K}(\mathscr{L}\mathscr{K})^j\mathscr{L} \subseteq \mathscr{L}$, a contradiction.

So it remains to show that $(Y, v)$ is admissible: if $(y, z) \in (Y \times Y) \cap \mathscr{L}^n$ and $(y, y) \in \mathscr{L}^{(1)}$ or $(z, z) \in \mathscr{L}^{(1)}$, then $(y, y)$, $(z, z) \in \mathscr{L}^{(1)}$, and thus $v(y) = 0 = (-1)^n v(z)$; otherwise we have $\{y, z\} = \{x_0, y_0\}$ and so we have $(y, z) \in \mathscr{K} \backslash \mathscr{L}$ which together with $(y, z) \in \mathscr{L}^n$ implies $n \equiv 0(2)$ (cf. (5.7f). So we have to show $v(x_0) = v(y_0)$. But either $v(x_0) = v(y_0) = 1$ or there exists $k \in \mathbb{N}$ with, say, $(x_0, x_0) \in (\mathscr{L}\mathscr{K})^k\mathscr{L}$, in which case $(x_0, y_0) \in \mathscr{L}^{2a}$ for $a = n/2 \in \mathbb{N}$ implies $(x_0, y_0) \in \mathscr{K} \cap (\mathscr{L}\mathscr{K})^{k+a}\mathscr{L} \subseteq \mathscr{L}$, a contradiction.

Now assume $(y, z) \in (Y \times Y) \cap \mathscr{L}^m\mathscr{K}(\mathscr{L}\mathscr{K})^k\mathscr{L}^j$ for some $m, k, j \in \mathbb{N}$. We have to show that $(-1)^m v(y) + (-1)^j v(z) \geqslant 0$. If $v(y) = v(z) = 0$, this is clear. It is also clear, if $y = z$, since $v(y) \geqslant 0$ for all $y \in Y$ and since $m \equiv j \equiv 1$ implies $(y, y) \in (\mathscr{L}\mathscr{K})^{((m+1)/2)+k+((j-1)/2)} \cdot \mathscr{L}$ and thus $v(y) = 0$. So we may assume $y \neq z$, $(y, y) \notin \mathscr{L}^{(1)}$, say $y = x_0$, $v(y) = 1$, and $m \equiv 1(2)$. If $(z, z) \in \mathscr{L}^{2n+1}$, this implies $(y, y) \in \mathscr{L}^m\mathscr{K}(\mathscr{L}\mathscr{K})^k\mathscr{L}^j\mathscr{L}^{2n+1}\mathscr{L}^j \mathscr{K}(\mathscr{L}\mathscr{K})^k\mathscr{L}^m \subseteq (\mathscr{L}\mathscr{K})^{((m+1)/2)+k+j+n+1+k+((m-1)/2)}\mathscr{L}$ in contradiction to $v(y) = 1$. So we have $z = y_0 \neq x_0$. If $j \equiv 1(2)$ we get $(x_0, y_0) \in \mathscr{K} \cap (\mathscr{L}\mathscr{K})^{((m+1)/2)+k+((j-1)/2)}\mathscr{L} \subseteq \mathscr{L}$, a contradiction, so we have $j \equiv 0(2)$ and we may assume $(y_0, y_0) \in (\mathscr{L}\mathscr{K})^a\mathscr{L}$ for some $a \in \mathbb{N}$. But this implies now $(x_0, y_0) \in \mathscr{K} \cap (\mathscr{L}\mathscr{K})^{((m+1)/2)+k+(j/2)+a} \cdot \mathscr{L} \subseteq \mathscr{L}$, again a contradiction. So $(Y, v)$ is indeed admissible.

*Remark.* In general, for a $\mathscr{K}$-closed relation $\mathscr{L} \subseteq \mathscr{K}$ we cannot derive $\mathscr{L} = \mathscr{L}_v^{\mathscr{K}}$ for some appropriate $v \in W_{\mathscr{L}}^{\mathscr{K}}$ from the representation $\mathscr{L} = \bigcap_{v \in W_{\mathscr{L}}^{\mathscr{K}}} \mathscr{L}_v^{\mathscr{K}}$. A counterexample is the following:

Let $Z$ be a partially ordered set such that there is not strictly monotonous map $f\colon Z \to \mathbb{R}$, e.g., because $Z$ is linearly ordered and of a cardinality which exceeds that of $\mathbb{R}$. Put $X = \{z^+, z^- \mid z \in Z\}$, $\mathscr{L} = \{(z^+, z^-) \mid z \in Z\}$, and $\mathscr{K} = \{(z^+, y^-) \mid z, y \in Z, z \geqslant y\}$, then $v\colon X \to \mathbb{R}$ is in $W_{\mathscr{L}}^{\mathscr{K}}$ if and only if $v(z^+) = -v(z^-)$ and $v(z^+) \geqslant v(y^+)$ for $z \geqslant y$. Thus $\mathscr{L} = \bigcap_{v \in W_{\mathscr{L}}^{\mathscr{K}}} \mathscr{L}_v^{\mathscr{K}}$, but $\mathscr{L} \neq \mathscr{L}_v^{\mathscr{L}}$ for all $v \in W_{\mathscr{L}}^{\mathscr{K}}$.

**(5.9)** As a first corollary we state: if $\mathscr{L} \subsetneqq \mathscr{K}$ is $\mathscr{K}$-closed and $\operatorname{supp} \mathscr{L} = \operatorname{supp} \mathscr{K} = X$, then $\mathscr{L}$ is a maximal $\mathscr{K}$-closed proper subrelation of $\mathscr{K}$ if and only if $\dim W_{\mathscr{L}}/W_{\mathscr{K}} = 1$.

*Proof.* If $\dim W_{\mathscr{L}}/W_{\mathscr{K}} = 1$ then $\mathscr{L}$ is maximal, since $\mathscr{L} \subsetneqq \mathscr{L}_1 \subsetneqq \mathscr{K}$ for some $\mathscr{K}$-closed relation $\mathscr{L}_1$ implies $W_{\mathscr{L}} \supsetneqq W_{\mathscr{L}_1} \supsetneqq W_{\mathscr{K}}$ by (5.7f). Vice versa, assume $\mathscr{L}$ to be maximal. It follows from (4.8), that any $v \in W_{\mathscr{L}}^{\mathscr{K}}$ is either in $W_{\mathscr{K}}$ or satisfies $v(x) + v(y) > 0$ for all $(x, y) \in \mathscr{K} \setminus \mathscr{L}$. Thus $(x, y) \in \mathscr{K} \setminus \mathscr{L}$ and $(y, y) \in (\mathscr{L}\mathscr{K})^k \mathscr{L}$ for some $k \in \mathbb{N}$ implies $(y, y) \in \mathscr{L}^{(1)}$, since it follows from the construction given in (5.8) that there exists some $v \in W_{\mathscr{L}}^{\mathscr{K}}$ with $v(x) = 1$ and $v(y) = 0$, so if $y = x_1, y_1, x_2, y_2, \ldots, x_{k+1}, y_{k+1} = y \in X$, $(x_1, y_1)$, $(x_2, y_2), \ldots$, $(x_{k+1}, y_{k+1}) \in \mathscr{L}$, and $(y_1, x_2)$, $(y_2, x_3), \ldots$, $(y_k, x_{k+1}) \in \mathscr{K}$ it follows from $v(y) = 0 = \sum_{i=1}^{k+1} (v(x_i) + v(y_i)) = v(y) + \sum_{i=1}^{k} (v(y_i) + v(x_{i+1})) + v(y)$ and $v(y_i) + v(x_{i+1}) \geqslant 0$ for all $i = 1, \ldots, k$ that $v(y_i) + v(x_{i+1}) = 0$ for all $i = 1, \ldots, k$ and thus $(y_1, x_2), \ldots, (y_k, x_{k+1}) \in \mathscr{L}$ and $(y, y) \in \mathscr{L}^{2k+1} \subseteq \mathscr{L}^{(1)}$.

Thus for any $(x, y) \in \mathscr{K} \setminus \mathscr{L}$ with $(x, x) \notin \mathscr{L}^{(1)}$ we can find some $v \in W_{\mathscr{L}}^{\mathscr{K}}$ with $v(X) \subseteq \{0, \pm 1\}$, $v(x) = 1$, and $v(y) \geqslant 0$. Note that $v(x) + v(y) > 0$ implies $v(a) + v(b) > 0$ and thus $v(a), v(b) \geqslant 0$ for any such $v$ and any $(a, b) \in \mathscr{K} \setminus \mathscr{L}$.

We claim that for any $(x, y)$, $(a, b) \in \mathscr{K} \setminus \mathscr{L}$ with $(a, a) \notin \mathscr{L}^{(1)}$ we have $(a, x) \in \mathscr{L}^{(0)} =: \bigcup_{n \in \mathbb{N}} \mathscr{L}^{2n}$ or $(a, y) \in \mathscr{L}^{(0)}$; choose $v, w \in W_{\mathscr{L}}^{\mathscr{K}}$ with $v(X)$, $w(X) \subseteq \{0, \pm 1\}$, $v(x) + v(y) > 0$, $w(a) = 1$, and $w(b) \geqslant 0$ and consider

$$u : X \to \mathbb{R} : z \mapsto 0 \qquad \text{if} \quad (z, x) \in \bigcup_{n \in \mathbb{N}} \mathscr{L}^n \text{ or } (z, y) \in \bigcup_{n \in \mathbb{N}} \mathscr{L}^n,$$

$$\mapsto v(z) + w(z) \qquad \text{otherwise.}$$

One easily verifies that $u(z) \geqslant 0$ whenever $(z, z') \in \mathscr{K} \setminus \mathscr{L}$ for some $z' \in X$ and that $u \in W_{\mathscr{L}}$, so $u \in W_{\mathscr{L}}^{\mathscr{K}}$. Thus $u(x) + u(y) = 0$ implies $u(a) + u(b) = 0$ which in turn implies $u(a) = 0 \neq v(a) + w(a)$ and hence $(a, x) \in \bigcup_{n \in \mathbb{N}} \mathscr{L}^n =$

$\mathscr{L}^{(0)} \cup \mathscr{L}^{(1)}$ or $(a, y) \in \mathscr{L}^{(0)} \cup \mathscr{L}^{(1)}$. But $(a, z) \in \mathscr{L}^{(1)}$ implies $w(z) = -w(a) = -1$, so we have $(a, x) \in \mathscr{L}^{(0)}$ or $(a, y) \in \mathscr{L}^{(0)}$.

It is now easy to see that $\dim W_{\mathscr{L}}/W_{\mathscr{K}} = 1$ or, more precisely, that $v \in W_{\mathscr{L}}$ and $v(x) + v(y) = 0$ for some fixed $(x, y) \in \mathscr{K}\backslash\mathscr{L}$ implies $v(a) + v(b) = 0$ for all $(a, b) \in \mathscr{K}\backslash\mathscr{L}$, i.e., $v \in W_{\mathscr{K}}$, since $(a, b) \in \mathscr{K}\backslash\mathscr{L}$ implies $(a, a) \in \mathscr{L}^{(1)}$ or $(a, x) \in \mathscr{L}^{(0)}$ or $(a, y) \in \mathscr{L}^{(0)}$. So $v(x) = v(y) = 0$ implies $v(a) = 0$ for all $(a, b) \in \mathscr{K}\backslash\mathscr{L}$, whereas $v(x) = -v(y) \neq 0$ implies $(x, x)$, $(y, y) \notin \mathscr{L}^{(1)}$ and thus it implies $(x, a) \in \mathscr{L}^{(0)}$ or $(x, b) \in \mathscr{L}^{(0)}$ as well as $(y, a) \in \mathscr{L}^{(0)}$ or $(y, b) \in \mathscr{L}^{(0)}$. In case $(x, a)$, $(y, b) \in \mathscr{L}^{(0)}$ or $(x, b)$, $(y, a) \in \mathscr{L}^{(0)}$ we get indeed $v(a) + v(b) = v(x) + v(y) = 0$, whereas in case, say, $(x, a)$, $(y, a) \in \mathscr{L}^{(0)}$ we get $(x, y) \in \mathscr{L}^{(0)}$ and thus $v(x) = v(y)$, in contradiction to $v(x) = -v(y) \neq 0$.

**(5.10)** From (5.9) one deduces easily: if $\mathscr{L} \subseteq \mathscr{K}$ is $\mathscr{K}$-closed, $\operatorname{supp} \mathscr{L} = \operatorname{supp} \mathscr{K}$, and $\dim W_{\mathscr{L}}/W_{\mathscr{K}} = 2$ then there exist precisely two $\mathscr{K}$-closed relations $\mathscr{L}_1$ and $\mathscr{L}_2$ with $\mathscr{L} \subsetneq \mathscr{L}_i \subsetneq \mathscr{K}$ $(i = 1, 2)$.

*Proof.* It follows immediately from (5.9) that there exists some $\mathscr{K}$-closed relation $\mathscr{L}_1$ with $\mathscr{L} \subsetneq \mathscr{L}_1 \subsetneq \mathscr{K}$ and thus $\dim W_{\mathscr{L}}/W_{\mathscr{L}_1} = \dim W_{\mathscr{L}_1}/W_{\mathscr{K}} = 1$. Thus $\mathscr{L}$ is a maximal $\mathscr{L}_1$-closed proper subrelation of $\mathscr{L}_1$ and $\mathscr{L}_1$ is a maximal $\mathscr{K}$-closed proper subrelation of $\mathscr{K}$. It follows from the above considerations that there exists some $v \in W_{\mathscr{L}}^{\mathscr{K}}\backslash W_{\mathscr{L}_1}$ and some $v_1 \in W_{\mathscr{L}_1}^{\mathscr{K}}\backslash W_{\mathscr{K}}$ with $v(X)$, $v_1(X) \subseteq \{0, \pm 1\}$. Put $c = \min((v(a) + v(b))/(v_1(a) + v_1(b)) \mid (a, b) \in \mathscr{K}\backslash\mathscr{L}_1)$ which exists and is nonnegative since $v(a) + v(b) \in \{0, 1, 2\}$ and $v_1(a) + v_1(b) \in \{1, 2\}$ for all $(a, b) \in \mathscr{K}\backslash\mathscr{L}_1$—so $c \in \{0, \frac{1}{2}, 1, 2\}$—and put $v_2 = v - cv_1$. Then $v_2 \in W_{\mathscr{L}}^{\mathscr{K}}\backslash W_{\mathscr{L}_1}$ and $v_2(a) + v_2(b) = 0$ for some $(a, b) \in \mathscr{K}\backslash\mathscr{L}_1$. Thus $\mathscr{L}_2 = \{(a, b) \in \mathscr{K} \mid v_2(a) + v_2(b) = 0\}$ is $\mathscr{K}$-closed and satisfies $\mathscr{L} \subsetneq \mathscr{L}_2 \subsetneq \mathscr{K}$ and $\mathscr{L}_1 \cap \mathscr{L}_2 = \mathscr{L}$, in particular $\mathscr{L}_1 \neq \mathscr{L}_2$. So it remains to show that there is no further $\mathscr{K}$-closed relation $\mathscr{L}_3$ with $\mathscr{L} \subsetneq \mathscr{L}_3 \subsetneq \mathscr{K}$. Otherwise choose some $v_3 \in W_{\mathscr{L}_3}^{\mathscr{K}}\backslash W_{\mathscr{K}}$. Since $W_{\mathscr{L}} = W_{\mathscr{K}} \oplus \langle v_1 \rangle \oplus \langle v_2 \rangle$ we have $w \in W_{\mathscr{K}}$, $c_1$, $c_2 \in \mathbb{R}$ with $v_3 = w + c_1 v_1 + c_2 v_2$. For $(a, b) \in \mathscr{L}_i\backslash\mathscr{L} = \mathscr{L}_i\backslash\mathscr{L}_3 \subseteq \mathscr{K}$ $(i = 1, 2)$ we get

$$\begin{aligned} 0 < v_3(a) + v_3(b) &= w(a) + w(b) + c_1(v_1(a) + v_1(b)) + c_2(v_2(a) + v_2(b)) \\ &= c_j(v_j(a) + v_j(b)) \end{aligned}$$

and thus $c_j > 0$ if $\{1, 2\} = \{i, j\}$. But this implies that for $(a, b) \in \mathscr{L}_3\backslash\mathscr{L} = \mathscr{L}_3\backslash(\mathscr{L}_1 \cup \mathscr{L}_2) \subseteq \mathscr{K}$ we have $0 = v_3(a) + v_3(b) = c_1(v_1(a) + v_1(b)) + c_2(v_2(a) + v_2(b)) > 0$, a contradiction.

We will use this result in the next section to construct boundary maps between cochaingroups defined on some set of relations $\mathscr{L} \subseteq X \times X$ with $\operatorname{supp} \mathscr{L} = X$ and $W_{\mathscr{L}}$ of some fixed finite dimension.

**(5.11)** Still another application of our analysis is the following result. For a $\mathscr{K}$-closed relation $\mathscr{L} \subseteq \mathscr{K}$ the following statements are equivalent:

(i) for any $\mathscr{K}$-closed relation $\mathscr{L}' \subseteq \mathscr{K}$ with $\mathscr{L} \subseteq \mathscr{L}'$ and $\operatorname{supp} \mathscr{L}' = \operatorname{supp} \mathscr{K}$ one has $\mathscr{L}' = \mathscr{K}$;

(ii) for any $\mathscr{K}$-closed relation $\mathscr{L}' \subseteq \mathscr{K}$ with $\mathscr{L} \subseteq \mathscr{L}'$ one has $\mathscr{L}' = \mathscr{K} \cap (\operatorname{supp} \mathscr{L}' \times \operatorname{supp} \mathscr{L}')$;

(iii) for any $\mathscr{K}$-closed relation $\mathscr{L}' \subseteq \mathscr{K}$ with $\mathscr{L} \subseteq \mathscr{L}'$ the restriction map $W_{\mathscr{K}} \to W_{\mathscr{L}'}: v \mapsto v|_{\operatorname{supp} \mathscr{L}'}$ is surjective.

In particular, if $\mathscr{L} \subseteq \mathscr{K}$ is an arbitrary $\mathscr{K}$-closed relation and if $\mathscr{L}_1 \subseteq \mathscr{K}$ is a $\mathscr{K}$-closed relation with $\mathscr{L} \subseteq \mathscr{L}_1$, $\operatorname{supp} \mathscr{L}_1 = \operatorname{supp} \mathscr{K}$, and $+\infty > \dim W_{\mathscr{L}_1} \geqslant \dim W_{\mathscr{L}'}$ for all $\mathscr{K}$-closed relations $\mathscr{L}' \subseteq \mathscr{K}$ with $\mathscr{L} \subseteq \mathscr{L}'$ and $\operatorname{supp} \mathscr{L}' = \operatorname{supp} \mathscr{K}$, then the restriction map $W_{\mathscr{L}_1} \to W_{\mathscr{L}}$ is surjective and $\mathscr{L}' = \mathscr{L}_1 \cap (\operatorname{supp} \mathscr{L}' \times \operatorname{supp} \mathscr{L}')$ for any $\mathscr{K}$-closed relation $\mathscr{L}'$ with $\mathscr{L} \subseteq \mathscr{L}'' \subseteq \mathscr{L}_1$.

*Proof.* W.l.o.g. assume $X = \operatorname{supp} \mathscr{K}$ and put $Y = \operatorname{supp} \mathscr{L}$.

(i) $\Rightarrow$ (ii). W.l.o.g. assume $\mathscr{L} = \mathscr{L}'$. If $a, b \in Y$ and $(a, b) \in \mathscr{K} \backslash \mathscr{L}$, then we can find some $v$: $Y \to \{0, \pm 1\}$ with $v \in W_{\mathscr{L}}$, $v(x) + v(y) \geqslant 0$ for all $(x, y) \in \mathscr{K} \cap (Y \times Y)$ and $v(a) + v(b) > 0$. Now put

$$X_1 =: \{x \in X \backslash Y \mid \text{there is some } y \in Y \text{ with } (x, y) \in \mathscr{K} \text{ and } v(y) = -1\},$$

$$X_0 =: \{x \in X \backslash (Y \cup X_1) \mid \text{there is some } y \in Y \text{ with } (x, y) \in \mathscr{K} \text{ and } v(y) = 0 \text{ or some } z \in X \backslash (Y \cup X_1) \text{ with } (x, z) \in \mathscr{K}\},$$

$$X_{-1} = X \backslash (Y \cup X_0 \cup X_1),$$

and

$$w: X \to \{0, \pm 1\}: x \mapsto \begin{cases} v(x) & \text{if } x \in Y, \\ i & \text{if } x \in X_i. \end{cases}$$

Using $\mathscr{K} \subseteq X \times X \backslash (X_0 \times X_{-1} \cup X_{-1} \times X_0)$ one easily verifies that $w \in W_{\mathscr{L}}^{\mathscr{K}}$, so $\mathscr{L}' = \{(x, y) \in \mathscr{K} \mid w(x) + w(y) = 0\}$ is $\mathscr{K}$-closed and satisfies $\mathscr{L} \subseteq \mathscr{L}' \subsetneqq \mathscr{K}$ and $\operatorname{supp} \mathscr{L}' = X$, a contradiction. Thus our assumption implies indeed $\mathscr{L} = \mathscr{K} \cap (Y \times Y)$.

(ii) $\Rightarrow$ (iii). W.l.o.g. we may assume $\mathscr{L}' = \mathscr{L}$ and $X = Y \cup \{x\}$. Assume $(x, y) \in \mathscr{K}$ for some $y \in Y$ which is contained in some bipartite component of $\mathscr{L}$. We have to show that $(x, y') \in \mathscr{K}$ implies $y' \in Y$ and $(y, y') \in \mathscr{L}^{(0)}$. Consider

$$v: X \to \{0, \pm 1\}: z \mapsto \begin{cases} +1 & \text{if } z = x, \\ -(-1)^n & \text{if } (y, z) \in \mathscr{L}^n, \\ 0 & \text{otherwise.} \end{cases}$$

Using $\mathscr{K}\backslash\mathscr{L}\subseteq X\times\{x\}\cup\{x\}\times X$ one easily verifies that $v\in W_{\mathscr{L}}^{\mathscr{K}}$ and $(x, y)\in\mathscr{L}'=\{(a, b)\in\mathscr{K}\mid v(a)+v(b)=0\}$. Thus $\operatorname{supp}\mathscr{L}'=X$ and hence $\mathscr{L}'=\mathscr{K}$, so $(x, y')\in\mathscr{K}$ implies $v(x)+v(y')=0$, i.e., $(y, y')\in\mathscr{L}^{(0)}$.

(iii) $\Rightarrow$ (i): This follows easily from (5.7f). Q.E.D.

**(5.12)** Let us now return to the consideration of a metric space $X$. It follows from the trivial part of (5.8) that for $f, g\in P_X$ the relation $\mathscr{K}_g$ is $\mathscr{K}_f$-closed, since $\mathscr{K}_g$ contains $\mathscr{L}_{g-f}^{\mathscr{K}_f}$ and since $(x, y)\in\mathscr{K}_g$ implies $(g(x)-f(x))+(g(y)-f(y)) = xy-f(x)-f(y)\leqslant 0$. Vice versa, if $\mathscr{L}\subseteq\mathscr{K}=\mathscr{K}_f$ is symmetric and if $\mathscr{L}$ is finite or if there exists some $\varepsilon>0$ with $xy_{\mathscr{K}}\geqslant xy+\varepsilon$ for all $(x, y)\notin\mathscr{K}$ (in particular, if $\mathscr{K}_f=\mathscr{K}_f^{\varepsilon}$ for some $\varepsilon>0$), then $\mathscr{L}$ is $\mathscr{K}$-closed if and only if $\mathscr{L}=\bigcap_{g\in P_X,\,\mathscr{L}\subseteq\mathscr{K}_g}\mathscr{K}_g$ since $xy_{\mathscr{L}}=xy$ for some $x, y\in X$ implies the existence of some $n\in\mathbb{N}$ and some $x=x_1, x_2,\ldots, x_n, y_1,\ldots, y_n=y_0=y\in X$ with $(x_1, y_1),\ldots, (x_n, y_n)\in\mathscr{L}\subseteq\mathscr{K}$ and with $\sum_{i=1}^n x_iy_i-\sum_{i=2}^n x_iy_{i-1}=xy$ or with $\sum_{i=1}^n x_iy_i-\sum_{i=2}^n x_iy_{i-1}<xy+\varepsilon$, respectively, which implies $x_iy_{i-1\,\mathscr{K}}=x_iy_{i-1}$ or $x_iy_{i-1\,\mathscr{K}}<x_iy_{i-1}+\varepsilon$, respectively, and thus $(x_i, y_{i-1})\in\mathscr{K}$ for all $i=1,\ldots, n$ which in turn implies $(x_1, y_0)=(x, y)\in\mathscr{L}$ for a $\mathscr{K}$-closed relation $\mathscr{L}$.

In particular, if $\mathscr{L}\subseteq\mathscr{K}$ is symmetric and if $\mathscr{L}$ is finite or $\mathscr{L}$ is countable and there exist some $\varepsilon>0$ with $xy_{\mathscr{K}}\geqslant xy+\varepsilon$ for all $(x, y)\notin\mathscr{K}$, then $\mathscr{L}$ is $\mathscr{K}$-closed if and only if it is of the form $\mathscr{L}=\mathscr{K}_g$ for some $g\in P_X$.

**(5.13)** Now define for any symmetric relation $\mathscr{K}\subseteq X\times X$ its rank, denoted by $\operatorname{rk}\mathscr{K}\in\mathbb{N}\cup\{\infty\}$, as the number of bipartite connected components of $\mathscr{K}$ or, equivalently, as the dimension of the vector space $W_{\mathscr{K}}$ and let $\dim\mathscr{K}$ denote the supremum of all numbers $\operatorname{rk}\mathscr{L}$, where $\mathscr{L}$ runs through all $\mathscr{K}$-closed relations $\mathscr{L}\subseteq\mathscr{K}$. Note that $\dim\mathscr{K}$ can be defined equivalently as the supremum of all numbers $\operatorname{rk}\mathscr{L}$, where $\mathscr{L}$ runs through all $\mathscr{K}$-closed relations $\mathscr{L}\subseteq\mathscr{K}$ with $\operatorname{supp}\mathscr{L}=\operatorname{supp}\mathscr{K}$ as well as the supremum of the numbers of connected components of $\mathscr{K}$-closed relations $\mathscr{L}\subseteq\mathscr{K}$ with $\mathscr{L}\cap\{(x, x)\mid x\in X\}=\varnothing$ as well as the supremum of all $n\in\mathbb{N}$ for which there exists some $x_1,\ldots, x_n, x_{-1},\ldots, x_{-n}\in X$ with $\#\{x_1,\ldots, x_n, x_{-1},\ldots, x_{-n}\}=2n$ such that $\mathscr{L}=\{(x_i, x_{-i})\mid i\in\{\pm 1,\ldots,\pm n\}\}$ is $\mathscr{K}$-closed.

In particular, if $\mathscr{K}=\mathscr{K}_f$ for some $f\in P_X$ and if $\dim\mathscr{K}_f\geqslant n\in\mathbb{N}$, then there exists some $g\in P_X$ with $\mathscr{K}_g\subseteq\mathscr{K}_f$ and $\operatorname{rk}\mathscr{K}_g=n$: just pick some $\mathscr{K}$-closed $\mathscr{L}=\{(x_i, x_{-i})\mid i\in\{\pm 1,\ldots,\pm n\}\}$ as above and pick some $g\in P_X$ with $\mathscr{K}_g=\mathscr{L}$.

**(5.14)** Note that $\dim\mathscr{K}=0$ if and only if $(x, y)\in\mathscr{K}$ implies $(x, x)\in\mathscr{K}$ and that $\dim\mathscr{K}\leqslant 1$ if and only if $x, y, z\in\operatorname{supp}\mathscr{K}$ and $(x, x)$, $(y, y)$, $(x, y)\notin\mathscr{K}$ implies $(x, z)\in\mathscr{K}\Leftrightarrow(y, z)\in\mathscr{K}$. In particular, if $(x, x)$, $(y, y)\in\mathscr{K}$ implies $x=y$—as is the case for $\mathscr{K}=\mathscr{K}_f$ for some $f\in P_X$—then $\dim\mathscr{K}\leqslant 1$ if and only if $\mathscr{K}$ is completely multipartite on its support.

Similarly, one has $\dim \mathscr{K} \leqslant 2$ for some symmetric relation $\mathscr{K} \subseteq X \times X$ for which $(x, x)$, $(y, y) \in \mathscr{K}$ implies $x = y$ if and only if $(x_1, x_{-1})$, $(x_2, x_{-2})$, $(x_3, x_{-3}) \in \mathscr{K}$ and $\#\{x_1, x_{-1}, x_2, x_{-2}, x_3, x_{-3}\} = 6$ implies the existence of a permutation $\alpha$ of $I =: \{\pm 1, \pm 2, \pm 3\}$ with $\alpha(-i) = -\alpha(i)$ $(i \in I)$ such that for $y_i = x_{\alpha(i)}$ one has $(y_{-1}, y_2)$, $(y_{-2}, y_1) \in \mathscr{K}$ or $(y_{-1}, y_2)$, $(y_{-2}, y_3)$, $(y_{-3}, y_1) \in \mathscr{K}$ or $(y_1, y_2)$, $(y_{-1}, y_2)$, $(y_3, y_{-2})$, $(y_{-3}, y_{-2}) \in \mathscr{K}$. A global characterization of 2-dimensional relations which corresponds to the global characterization of the 1-dimensional ones as those which are essentially multipartite does not seem to be that easy, though it is evident that $\dim \mathscr{K} = n$, $(x, y) \in \mathscr{K}$, and $x \neq y$ implies $\dim \mathscr{K}^{(x,y)} \leqslant n - 1$ for $\mathscr{K}^{(x,y)} =: \{(a, b) \in \mathscr{K} \mid \mathscr{K} \cap \{(a, x), (b, x), (a, y), (b, y)\} = \varnothing\}$, whereas $\dim \mathscr{K}^{(x,y)} \leqslant n - 1$ for all $(x, y) \in \mathscr{K}$ with $x \neq y$ does not necessarily imply $\dim \mathscr{K} \leqslant n$.

But at least we can state that we have $\dim \mathscr{K} = \operatorname{rk} \mathscr{K} = n < \infty$ if and only if $(x, x) \in \mathscr{K}^{(1)}$ implies $(x, x) \in \mathscr{K}$ and any bipartite connected component of $\mathscr{K}$ is completely bipartite if and only if $\operatorname{rk} \mathscr{K} = n < \infty$ and any $\mathscr{K}$-closed relation $\mathscr{L} \subseteq \mathscr{K}$ is of the form $\mathscr{L} = \mathscr{K} \cap (Y \times Y)$ for some $Y \subseteq X$ (namely, $Y = \operatorname{supp} \mathscr{L}$).

So we have $\dim \mathscr{K} = n < \infty$ if and only if any bipartite $\mathscr{K}$-closed relation $\mathscr{L} \subseteq \mathscr{K}$ with $\operatorname{rk} \mathscr{L} = n$ satisfies $\mathscr{L} = \mathscr{L}^{(1)}$ and we have

$$\dim \mathscr{K} = \sup(\operatorname{rk} \mathscr{L} \mid \mathscr{L} \subseteq \mathscr{K} \text{ is } \mathscr{K}\text{-closed and satisfies } \mathscr{L} = \mathscr{L}^{(1)} \text{ and } \mathscr{L} \cap \mathscr{L}^{(0)} = \varnothing).$$

**(5.14)** Let us finally define the combinatorial dimension $\dim_{\text{comb}} X$ of a metric space $X$ to be the supremum of all the dimensions of all $\mathscr{K}_f$ with $f \in P_X$. Since $f, g \in P_X$ and $f \leqslant g$ implies that $\mathscr{K}_g$ is $\mathscr{K}_f$-closed and contained in $\mathscr{K}_f$, we also have

$$\dim_{\text{comb}} X = \sup(\dim \mathscr{K}_f \mid f \in T_X).$$

To simplify our notations let us also write $\dim f$ and $\operatorname{rk} f$ instead of $\dim \mathscr{K}_f$ and $\operatorname{rk} \mathscr{K}_f$, respectively.

We are now ready to prove Theorem 9 in the following more complete form:

THEOREM 9′. *For a metric space $X$ the following conditions are equivalent*:

(i) $\dim_{\text{comb}} X < n$;

(i′) $\dim_{\text{comb}} Y < n$ *for all* $Y \subseteq X$;

(ii) $\dim f < n$ *for all* $f \in T_X$;

(ii′) $\dim f < n$ *for all* $f \in P_X$;

(iii) $\operatorname{rk} f < n$ *for all* $f \in P_X$;

(iii′) $\operatorname{rk} f < n$ *for all* $f \in P_X$ *with* $\#\mathscr{K}_f = 2n$;

(iv) *for all* $x_1, x_{-1}, x_2, x_{-2}, ..., x_n, x_{-n} \in X$ *there exists a permutation* $\alpha$ *of* $\{\pm 1, ..., \pm n\} = I$ *with* $\alpha \neq -\mathrm{Id}_I$ *and* $\sum_{I \in I} x_i x_{-i} \leqslant \sum_{i \in I} x_i x_{\alpha(i)}$;

(v) $\dim T_Y < n$ *for all finite* $Y \subseteq X$;

(v′) $\dim T_Y < n$ *for all* $Y \subseteq X$ *with* $\#Y = 2n$;

(vi) $\dim_{\mathrm{comb}} T_X < n$;

(vi′) $\dim_{\mathrm{comb}} T_Y < n$ *for all* $Y \subseteq X$.

*Proof.* The equivalence of (i), (ii), and (ii′) follows directly from the definitions and the above considerations. The implication (i′) ⇒ (i) is trivial, the implication (ii″) ⇒ (i′) follows from the fact that any $g \in T_Y$ can be extended to some $f \in T_X$ and thus to some $f' \in P_X$ with $\mathscr{K}_g = \mathscr{K}_{f'}$, e.g.,

$$\begin{aligned} f'(x) &= f(x) && \text{for} \quad x \in Y, \\ &= f(x) + 1 && \text{for} \quad x \in X \backslash Y. \end{aligned}$$

(ii′) ⇒ (iii) and (iii) ⇒ (iii′) are trivial, (iii′) ⇒ (ii′) follows from the fact that $\dim f \geqslant n$ for some $f \in P_X$ implies the existence of some $g \in P_X$ with $\#\mathscr{K}_g = 2n$ and $\operatorname{rk} g = n$ by (5.13).

(iii′) ⇔ (iv) follows from (5.3). To prove (i) ⇒ (vi) assume $f_1, f_{-1}, ..., f_n, f_{-n} \in T_X$ and $I = \{\pm 1, ..., \pm n\}$. W.l.o.g. we may assume that for all $i \in I \backslash \{-n\}$ we have $f_i = h_{x_i}$ for some $x_i \in X$. But this implies

$$\begin{aligned} \sum_{i \in I} \|f_i, f_{-i}\| &= \sup\left(\sum_{i \in I} x_i x_{-i} - 2f_{-n}(x_{-n}) \,\middle|\, x_{-n} \in X\right) \\ &\leqslant \sup\left(\sum_{i \in I} x_i x_{\alpha(i)} - 2f_{-n}(x_{-n}) \,\middle|\, x_{-n} \in X, \alpha \neq -\mathrm{Id}_I\right) \\ &= \sup\left(\sum_{i \in I} \|f_i, f_{\alpha(i)}\| \,\middle|\, \alpha \neq -\mathrm{Id}_I\right). \end{aligned}$$

The implications (i′) ⇒ (vi′) ⇒ (vi) ⇒ (i) are now trivial.

(iii) ⇒ (v) follows from the fact that for finite $Y$ the space $T_Y$ is the union of the finitely many closed subsets $\bar{\mathscr{K}} =: \{f \in T_Y \mid \mathscr{K} \subseteq \mathscr{K}_f\}$, where $\mathscr{K}$ runs through all relations $\mathscr{K} \subseteq Y \times Y$ for which there exist some $f \in T_Y$ with $\mathscr{K} = \mathscr{K}_f$, and that for $f \in T_Y$ and $\mathscr{K} = \mathscr{K}_f$ the map $\bar{\mathscr{K}} \to W_{\mathscr{K}}$: $g \mapsto g - f$ is injective and its image is a compact convex subset of $W_{\mathscr{K}}$ which contains a neighbourhood of $0 \in W_{\mathscr{K}}$. So we have $\dim \bar{\mathscr{K}} = \dim W_{\mathscr{K}} = \operatorname{rk} f < n$ and thus we have $\dim T_Y \leqslant \max(\operatorname{rk} f \mid f \in T_Y) < n$.

Since (v) $\Rightarrow$ (v′) is trivial, it remains to show that (v′) implies (iv). So assume $Y=\{x_1, x_{-1},..., x_n, x_{-n}\}\subseteq X$ and $\sum_{i\in I} x_i x_{-i} > \sum_{i\in I} x_i x_{\alpha(i)}$ for all permutations $\alpha \neq -\mathrm{Id}_I$ of $I=\{\pm 1,..., \pm n\}$. By (5.3) this implies $\# Y=2n$ and the existence of some $f\in T_Y$ with $\mathscr{K}_f=\mathscr{K}=: \{(x_i, x_{-i}) \mid i\in I\}$ and thus $\dim T_Y \geqslant \dim \bar{\mathscr{K}} = \mathrm{rk}\,\mathscr{K} = n$, a contradiction.

**(5.15)** Next we show: if $X_1$ and $X_2$ are metric spaces with $\dim_{\mathrm{comb}} X_\nu = n_\nu$ $(\nu=1,2)$, then $\dim_{\mathrm{comb}} X_1\times X_2 \leqslant n_1+n_2$. Moreover, if $X_1$ and $X_2$ are fully spread, then $\dim_{\mathrm{comb}} X_1\times X_2 = n_1+n_2$.

*Proof.* Let $n=n_1+n_2+1$ and choose a pair $(x_i^1, x_i^2)\in X_1\times X_2$ for any $i\in I=\{\pm 1,..., \pm n\}$. Let $I_\nu=\{i\in I \mid \sup(x_i^1 x_{-i}^1,\ x_i^2 x_{-i}^2)=x_i^\nu x_{-i}^\nu\}$ $(\nu=1,2)$. Then $I_\nu=-I_\nu$ and $I_1\cup I_2=I$. Thus $\# I_1 > 2n_1$ or $\# I_2 > 2n_2$. But if $\# I_\nu > 2n_\nu$, then there exists some permutation $\alpha_\nu$ of $I_\nu$ with $\alpha_\nu \neq -\mathrm{Id}_{I_\nu}$ and $\sum_{i\in I_\nu} x_i^\nu x_{-i}^\nu \leqslant \sum_{i\in I_\nu} x_i^\nu x_{\alpha(i)}^\nu$. Extending $\alpha_\nu$ by $-\mathrm{Id}_I$ on $I\backslash I_\nu$ we get a permutation $\alpha$ of $I$ with $\alpha\neq -\mathrm{Id}_I$ and $\sum_{i\in I}\sup(x_i^1 x_{-i}^1,\ x_i^2 x_{-i}^2)\leqslant \sum_{i\in I}\sup(x_i^1 x_{\alpha(i)}^1, x_i^2 x_{\alpha(i)}^2)$ which proves $\dim_{\mathrm{comb}} X_1\times X_2 < n = n_1+n_2+1$.

If $X_1$ and $X_2$ are fully spread then we may choose for given $x_1^\nu, x_{-1}^\nu,..., x_{n_\nu}^\nu, x_{-n_\nu}^\nu \in X_\nu$ $(\nu=1,2)$ with $\sum_{i\in I_\nu} x_i^\nu x_{-i}^\nu > \sum_{i\in I_\nu} x_i^\nu x_{\alpha_\nu(i)}^\nu$ for all permutations $\alpha_\nu$ of $I_\nu =: \{\pm 1,..., \pm n_\nu\}$ with $\alpha_\nu \neq -\mathrm{Id}_{I_\nu}$ some $x^\nu\in X_\nu$ with $x^\nu x_i^\nu + x^\nu x_j^\nu = x_i^\nu x_j^\nu$ if and only if $i+j=0$. Now put $m=n_1+n_2$ and consider the sequence $(y_1, z_1), (y_{-1}, z_{-1}),..., (y_m, z_m), (y_{-m}, z_{-m})\in X_1\times X_2$ defined by

$$y_i = x_i^1 \ \text{ if } |i|\leqslant n_1, \qquad\text{and}\qquad z_i = x^2 \ \text{ if } i\leqslant n_1,$$
$$\ \ = x^1 \ \text{ if } |i| > n_1, \qquad\qquad\qquad\ \ = x_{i-n_1}^2 \ \text{ if } i > n_1,$$
$$= x_{i+n_1}^2 \ \text{ if } i < -n_1,$$

and the map $f$: $\{(y_i, z_i) \mid i=\pm 1,..., \pm m\}\to \mathbb{R}$ defined by

$$\begin{aligned} f(y_i, z_i) &= x^1 x_i^1 && \text{if } |i|\leqslant n_1,\\ &= x^2 x_{i-n_1}^2 && \text{if } i > n_1,\\ &= x^2 x_{i+n_1}^2 && \text{if } i < -n_1. \end{aligned}$$

Then one has $f(y_i, z_i)+f(y_j, z_j)\geqslant y_i y_j, z_i z_j$ for all $i, j\in\{\pm 1,..., \pm m\}$ and one has $f(y_i, z_i)+f(y_j, z_j)=\sup(y_i y_j, z_i z_j)$ if and only if $i+j=0$. Thus $f\in P_{\{(y_i, z_i)\mid i=\pm 1,..., \pm m\}}$ and $\mathrm{rk}\, f=m$ which implies $\dim_{\mathrm{comb}}\{(y_i, z_i) \mid i=\pm 1,..., m\}\geqslant m$ and thus $\dim_{\mathrm{comb}} X_1\times X_2\geqslant m = n_1+n_2$.

**(5.16)** Next we want to show a rather technical lemma; so assume $Y\subseteq X$, $f\in T_X$, $g=f|_Y\in T_Y$, $f\neq g^*$ (with $g^*(x)=\sup(xy-g(y) \mid y\in Y)$, cf. (1.11)), $\dim g<\infty$, and $\varepsilon>0$. We claim the existence of some $f'\in T_X$ with $f'|_Y=g$, $\|f, f'\|<\varepsilon$, and $\dim f' > \dim g$.

*Proof.* By (1.11) there exists some $x_1 \in X \setminus Y$ with $f(x_1) > g^*(x_1)$. Choose some $\eta > 0$ with $3\eta \leqslant \varepsilon$ and $3\eta \leqslant f(x_1) - g^*(x_1)$ and choose some $x_2 \in X$ with $f(x_1) + f(x_2) \leqslant x_1 x_2 + \eta$. Since $f(x_1) \geqslant 3\eta + g^*(x_1) \geqslant 3\eta + x_1 y - g(y)$ for all $y \in Y$, we have $x_2 \notin Y$. Now put $Z = Y \cup \{x_1, x_2\}$ and define

$$\begin{aligned} f'' : Z \to \mathbb{R} : z &\mapsto g(z) && \text{if } z \in Y, \\ &\mapsto f(x_2) + \eta && \text{for } z = x_2, \\ &\mapsto x_1 x_2 - f(x_2) - \eta && \text{for } z = x_1. \end{aligned}$$

Then we have $f''|_Y = g$, $f'' \in T_Z$, and $\mathscr{K}_{f''} = \mathscr{K}_g \cup \{(x_1, x_2), (x_2, x_1)\}$ since $f''(x_1) + f''(x_2) = x_1 x_2$, $f''(x_2) + f''(y) = f(x_2) + f(y) + \eta > x_2 y$ for $y \in Y$, and $f''(x_1) + f''(y) = x_1 x_2 - f(x_2) - \eta + g(y) \geqslant f(x_1) - \eta - \eta + g(y) \geqslant 3\eta + x_1 y - 2\eta > x_1 y$ for $y \in Y$. Thus $\dim f'' = \dim g + 1$. Now, using $\|f'', f|_Z\| = \sup(|f''(x_1) - f(x_1)|, |f''(x_2) - f(x_2)|) = \sup(|x_1 x_2 - f(x_2) - \eta - f(x_1)|, \eta) = (f(x_1) + f(x_2) - x_1 x_2) + \eta \leqslant 2\eta < \varepsilon$, and the last remark in (1.11) there exists some $f' \in T_X$ with $\|f', f\| \leqslant 2\eta < \varepsilon$ and $f'|_Z = f''$ which implies in particular $f'|_Y = g$ and $\dim f' \geqslant \dim f'' > \dim g$.

**(5.17)** Now remember that $T_X^0 = \{f \in T_X \mid \operatorname{supp} \mathscr{K}_f = X\}$. Generalizing the case $\dim_{\text{comb}} X \leqslant 1$ considered in (4.8) we claim that $T_X^0$ is dense in $T_X$ for any metric space $X$ of finite dimension. More generally, we claim that any $f \in T_X$ for which there exists some $\varepsilon > 0$ and some $n \in \mathbb{N}$ with $\dim f' \leqslant n$ for all $f' \in T_X$ with $\|f', f\| \leqslant \varepsilon$ is contained in $\overline{T_X^0}$.

*Proof.* For each $\eta > 0$ with $\eta \leqslant \varepsilon$ choose some $f' \in T_X$ with $\|f', f\| < \eta$ and $\dim f' = m =: \max(\dim f'' \mid f'' \in T_X$ and $\|f'', f\| < \eta)$.

Then there exists a finite subset $Y \subseteq X$ with $g = f'|_Y \in T_Y$ and $\dim g = \dim f'$. Now it follows from (5.16) and the maximality of $\dim f'$ that $f' = g^*$. But $g^* \in T_X^0$ if $g \in T_Y$, $g^* \in T_X$, and $\#Y < \infty$, so for each $\eta > 0$ we have some $f' = g^* \in T_X^0$ with $\|f', f\| < \eta$, i.e., we have $f \in \overline{T_X^0}$.

**(5.18)** Another consequence of the same argument is that for any $f \in T_X$ with $\mathscr{K}_f = \mathscr{K}_f^\varepsilon$ for some $\varepsilon > 0$ (and thus surely $f \in T_X^0$) one has $f = (f|_Y)^*$ for any $Y \subseteq X$ with $f|_Y \in T_Y$ and $\dim f = \dim f|_Y$, since $\mathscr{K}_f = \mathscr{K}_f^\varepsilon$ and $\|g, f\| \leqslant \varepsilon/2$ implies $\mathscr{K}_g \subseteq \mathscr{K}_f^\varepsilon = \mathscr{K}_f$ and thus $\dim g \leqslant \dim f$ for all $g \in T_X$ with $\|g, f\| \leqslant \varepsilon/2$. So we may choose $f = f'$ in the above argument.

**(5.19)** Finally we define a space $X$ to be strongly discrete if for any $f \in T_X$ there is some $\varepsilon > 0$ with $\mathscr{K}_f = \mathscr{K}_f^\varepsilon$. Spaces with this property will be studied extensively in the next section. Here we show: if $\dim_{\text{comb}} X = n < \infty$ and $xy \in \mathbb{N}$ for all $x, y \in X$, then $X$ is strongly discrete.

*Proof.* Assume $f \in T_X$ and choose a sequence $f_1, f_2, \ldots \in T_X^0$ with $\|f, f_\nu\| \leqslant 1/\nu$. Since $\dim f_\nu \leqslant n$ the relation $\mathscr{K}_{f_\nu}$ has at most $n$ connected components. Since $f_\nu(x) + f_\nu(y) \equiv 0 \bmod 1$ for all $(x, y) \in \mathscr{K}_{f_\nu}$ it follows that $\#\{e^{2\pi i f_\nu(x)} \mid x \in X\} \leqslant 2n$. This in turn implies

$$\#\{e^{2\pi i f(x)} \mid x \in X\} \leqslant 2n.$$

So there exists some $\varepsilon > 0$ such that $|f(x) + f(y) - n| < \varepsilon$ for some $n \in \mathbb{Z}$ implies $f(x) + f(y) = n$. So, in particular, $f(x) + f(y) \leqslant xy + \varepsilon$ implies $f(x) + f(y) = xy$. Q.E.D.

**(5.20)** I conjecture that $\dim_{\text{comb}} X < n$; $f_1, g_1, \ldots, f_m, g_m \in T_X$ and $\bigcap_{i \in J} \langle f_i, g_i \rangle_{T_X} \neq \varnothing$ for all $J \subseteq \{1, \ldots, m\}$ with $\# J \leqslant n$ implies

$$\bigcap_{i=1}^{m} \langle f_i, g_i \rangle_{T_X} \neq \varnothing.$$

Since $\langle f, g \rangle$ is convex in case $X$ is tree-like by (4.5), this conjecture is true for $n = 2$ in view of (2.5).

## 6. Strongly Discrete Spaces and Pseudo-Convex Polytopes

**(6.1)** Let $W$ be a real vector space. For any subset $T \subseteq W$ let $[T] =: \{\sum_{i=1}^n \lambda_i v_i \mid n \in \mathbb{N};\ \lambda_1, \ldots, \lambda_n \in \mathbb{R};\ v_1, \ldots, v_n \in T;\ \sum_{i=1}^n \lambda_i = 1;\ \lambda_i \geqslant 0$ for all $i = 1, \ldots, n\}$ denote its convex hull and for $v, w \in W$ recall that $(v, w) =: \{\lambda v + (1 - \lambda) w \mid 0 < \lambda < 1\}$, so we have $(v, w) = \{w\}$ if $v = w$ and $(v, w) = [v, w] \setminus \{v, w\}$ otherwise. Now assume that $W$ is endowed with a map $\|\cdots\|\colon W \to \mathbb{R} \cup \{+\infty\}$ satisfying the usual conditions of a norm, i.e.,

$$\|v\| = 0 \Leftrightarrow v = 0,$$

$$\|v_1 + v_2\| \leqslant \|v_1\| + \|v_2\|,$$

and

$$\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$$

for all $v, v_1, v_2 \in W$, we $\lambda \in \mathbb{R}$.

The example we have in mind is of course $W = \mathbb{R}^X$ with $\|f\| = \sup(|f(x)| \mid x \in X)$.

A subset $P \subseteq W$ is defined to be pseudo-convex polytope (in $W$ with respect to $\|\cdots\|$) if it is a closed subset, satisfying the following conditions:

(P0) $f, g \in P \Rightarrow \|f-g\| < +\infty$;

(P1) for each $f \in P$ there is some $\varepsilon > 0$ such that $g \in P \backslash \{f\}$ and $\|f-g\| \leqslant \varepsilon$ implies $[f, f+(\varepsilon/\|f-g\|)(g-f)] \subseteq P$;

(P2) if $g \in (g_1, g_2) \subseteq P$ and $[g, f] \subseteq P$, then $[g_1, f] \subseteq P$;

(P3) if $f \in P$, then the vector space $W_f$ spanned by all $v \in W$ with $[f-v, f+v] \subseteq P$ has finite dimension;

(P4) for each $f \in P$ and $v \in W \backslash \{0\}$ there is some $\lambda > 0$ with $f+\lambda v \notin P$.

Note that (P2) is equivalent to

(P2′) $g \in (g_1, g_2) \subseteq P$ and $[g, f] \subseteq P$ implies $[g_1, g_2, f] \subseteq P$.

**(6.2)** If $X$ is a strongly discrete metric space and if $\operatorname{rk} f < \infty$ for all $f \in T_X$, then $P = T_X \subseteq W = \mathbb{R}^X$ is a pseudo-convex polytope.

*Proof.* (P0) follows from (1.6).

(P1) Assume $f \in T_X$ and choose some $\varepsilon > 0$ with $\mathscr{K}_f = \mathscr{K}_f^{2\varepsilon}$. Now assume $g \in T_X \backslash \{f\}$ and $\|f-g\| \leqslant \varepsilon$. It follows that $\mathscr{K}_g \subseteq \mathscr{K}_f^{2\varepsilon} = \mathscr{K}_f$ and thus $f+\lambda(g-f) \in T_X$ for all $\lambda \in \mathbb{R}$ with $0 \leqslant \lambda \leqslant \varepsilon/\|f-g\|$, since $(f(x)+\lambda(g(x)-f(x)))+(f(y)+\lambda(g(y)-f(y))) = xy+\lambda(g(x)+g(y)-xy)+(1-\lambda)(f(x)+f(y)+xy)$ equals $xy$ for $(x, y) \in \mathscr{K}_g$ and is larger than $xy$ for $(x, y) \in \mathscr{K}_f \backslash \mathscr{K}_g$, whereas for $(x, y) \notin \mathscr{K}_f$ it equals $f(x)+f(y)+\lambda(g(x)-f(x))+\lambda(g(y)-f(y))$ and so it is larger than $xy+2\varepsilon-2\lambda \cdot \|f-g\| \geqslant xy$.

(P2) If $g \in (g_1, g_2) \subseteq T_X$ and $[g, f] \subseteq T_X$, then $\mathscr{K}_{g_1} \cap \mathscr{K}_{g_2} = \mathscr{K}_g$ and $\operatorname{supp}(\mathscr{K}_g \cap \mathscr{K}_f) = X$. Thus $\operatorname{supp}(\mathscr{K}_f \cap \mathscr{K}_{g_1}) = X$ and therefore $[f, g_1] \subseteq T_X$.

(P3) If $f \in T_X$, $v \in \mathbb{R}^X$, and $f \pm v \in T_X$, then $v(x)+v(y) = 0$ for all $(x, y) \in \mathscr{K}_f$, i.e., we have $W_f \subseteq W_{\mathscr{K}_f}$ and thus we have $\dim W_f \leqslant \dim W_{\mathscr{K}_f} = \operatorname{rk} f < \infty$. More precisely, we have $W_f = W_{\mathscr{K}_f}$, since $\operatorname{rk} f < \infty$ implies $\|v\| < \infty$ for all $v \in V_{\mathscr{K}_f}$ and since $\mathscr{K}_f = \mathscr{K}_f^{2\varepsilon}$ implies $[f+v, f-v] \subseteq T_X$ for all $v \in W_{\mathscr{K}_f}$ with $\|v\| < \varepsilon$.

(P4) If $f \in T_X$, $v \in \mathbb{R}^X$, and $f+\lambda v \in T_X$ for all $\lambda > 0$, then $[(f+v)-v, (f+v)+v] \subseteq T_X$ and thus $v \in \mathscr{K}_{f+v}$. In particular, $v \neq 0$ implies $v(x) < 0$ for some $x \in X$ and so it implies $f+\lambda v \notin T_X$ for $\lambda = 1+f(x)/-v(x) > 0$, a contradiction.

**(6.3)** From now on assume $P \subseteq W$ to be an arbitrary pseudo-convex polytope with respect to some map $\|\cdots\|: W \to \mathbb{R} \cup \{+\infty\}$. We define the relation $\prec$ on $P \times P$ by $g \prec f \Leftrightarrow [g, f+\varepsilon(f-g)] \subseteq P$ for some $\varepsilon > 0$. Note that $f = (\varepsilon/(1+\varepsilon))\,g + (1/(1+\varepsilon))(f+\varepsilon(f-g)) \in (g, f+

$\varepsilon(f-g))$ (cf. Fig. 1a). The following properties of this relation are more or less obvious:

(R0) $f \prec f$ for all $f \in P$;

(R1) $g \prec f$ implies $f - g \in W_f$;

(R2) $g \prec f$ implies $W_g \subseteq W_f$ (cf. Fig. 1b);

(R3) $g \in (g_1, g_2) \subseteq P$ and $g \prec f$ implies $[g_1, g_2] \prec f$, in particular, $h \prec g$ and $g \prec f$ implies $h \prec f$ and, so, the relation $\asymp$ defined by $f \asymp g \Leftrightarrow f \prec g$ and $g \prec f$ is an equivalence relation on $P$ (cf. Figs. 1c, d);

(R4) $g_1 \prec f$ and $g_2 \prec f$ implies $[g_1, g_2] \prec f$ (cf. Fig. 1e);

(R5) for each $f \in P$ there exists some $\varepsilon > 0$ (namely, the $\varepsilon$ from (P1)) such that $g \in P$ and $\|f - g\| < \varepsilon$ implies $f \prec g$, in particular, the set $\tilde{f} =: \{g \in P \mid g \prec f\}$ is closed.

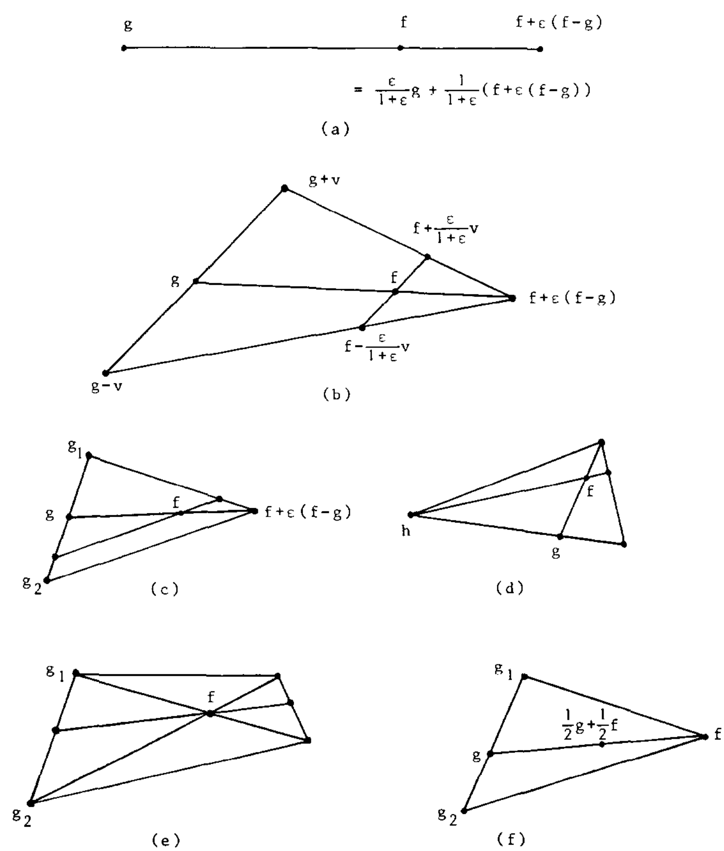



FIGURE 1

**(6.4)** Next we claim: for each $f \in P$ there is some $\varepsilon > 0$ such that $v \in W_f$ and $\|v\| < \varepsilon$ implies $f + v \in P$ and thus $f + v \asymp f$.

*Proof.* Let $v_1, \ldots, v_n \in W_f$ be a basis of $W_f$ such that $[f - v_i, f + v_i] \subseteq P$ for all $i = 1, \ldots, n$. It follows that $f \pm v_i \prec f$ for all $i = 1, \ldots, n$ and thus, using (R4), $f + \sum_{i=1}^n \lambda_i v_i \prec f$ for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n |\lambda_i| \leqslant 1$. But $\{\sum_{i=1}^n \lambda_i v_i \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \sum_{i=1}^n |\lambda_i| \leqslant 1\}$ contains a full $\varepsilon$-neighbourhood of $0$ in $W_f$. Q.E.D.

**(6.5)** As a corollary we derive:
If $[f, g] \subseteq P$ then $g \prec f$ if and only if $f - g \in W_f$; in particular, if $g \prec f$, then $g \asymp f$ if and only if $g - f \in W_g$ if and only if $W_f = W_g$ if and only if $W_f \subseteq W_g$.

*Proof.* We know from (R1) that $g \prec f$ implies $f - g \in W_f$. Vice versa, if $f - g \in W_f$, then $[f, f + \varepsilon(f - g)] \subseteq P$ for some $\varepsilon > 0$, so $[g, f] \subseteq P$ implies $[g, f + \varepsilon(f - g)] = [g, f] \cup [f, f + \varepsilon(f - g)] \subseteq P$, i.e., $g \prec f$.

**(6.6)** Next we claim:
Assume $f \in P$ and $\dim W_f = n$. Then $\tilde{f} = \{g \in P \mid g \prec f\}$ is a convex, compact polytope of dimension $n$, spanned by finitely many points in $W$, whose interior consists of all $g \in P$ with $g \asymp f$, in particular, $f$ is contained in its interior.

*Proof.* We know already from (R4) and (R5) that $\tilde{f}$ is closed and convex, we know from (R1) that $\tilde{f} - f =: \{g - f \mid g \in \tilde{f}\}$ is contained in $W_f$ and we know from (6.4) that $\tilde{f} - f$ contains a full $\varepsilon$-neighbourhood of 0 in $W_f$, so $\tilde{f}$ is of dimension $n$ and $f$ is contained in its interior.

Finally, $\tilde{f} - f$ contains no "half-line" $\{\lambda v \mid \lambda > 0\}$ $(v \in W \setminus \{0\})$ because of (P4), so—being closed and convex in the finite dimensional vectorspace $W_f$—it is compact. It is a polytope in the sense of pl-topology, since—by (P1) and (R3)—there exists for each $g \prec f$ some $\varepsilon > 0$ such that $h \prec f$, $h \neq g$, and $\|h - g\| < \varepsilon$ implies $[g, g + (\varepsilon/\|h - g\|)(h - g)] \prec f$, so the set $\tilde{f} - f$ is the union of finitely many simplices (cf. [26]) and, hence, it is the convex hull of finitely many points in $W_f$. Thus $\tilde{f}$ is a compact, convex polytope of dimension $n$ which contains $f$ in its interior $\tilde{f} \setminus \partial \tilde{f}$. The same holds for any $g \in P$ with $g \asymp f$, since $g \asymp f$ implies $\tilde{g} = \tilde{f}$. Vice versa, if $g \in P$ is contained in the interior $\tilde{f} \setminus \partial \tilde{f}$ of $\tilde{f}$, then $g \prec f$ and $W_f \subseteq W_g$ and, thus, $g \asymp f$. So the boundary $\partial \tilde{f}$ of $\tilde{f}$ consists precisely of those $g \prec f$ with $W_g \subsetneq W_f$.

**(6.7)** Now for each $n \in \mathbb{N}$ let $P_n$ denote the set $P_n =: \{f \in P \mid \dim W_f \leqslant n\}$. $P_n$ is closed because of (R5). It is a pseudo-convex polytope, since it inherits (P0), (P3), and (P4) directly from $P$. It inherits (P1), too, since $f, g \in P_n$, $0 \neq \|f - g\| < \varepsilon$, and $T =: [f, f + (\varepsilon/\|f - g\|)(g - f)] \subseteq P$

implies $T \prec g$ and thus $T \subseteq P_n$. And it satisfies (P2′), since $g \in (g_1, g_2) \subseteq P_n$ and $[g, f] \subseteq P_n$ implies $[g_1, g_2, f] \subseteq P$ as well as $[g_1, g_2, f] \prec \frac{1}{2}g + \frac{1}{2}f \in P_n$ and thus $[g_1, g_2, f] \subseteq P_n$ (cf. Fig. 1f). We claim that we can choose in each $\divideontimes$-equivalence class $\tilde{f}\backslash\partial\tilde{f}$ of dimension $n$ some $g = g_{\tilde{f}}$ and some $\varepsilon = \varepsilon_{\tilde{f}} > 0$ such that $B_{\tilde{f}} =: \{h \in P_n \,\|\, \|h - g\| \leqslant \varepsilon\}$ is contained in $\tilde{f} - \partial\tilde{f}$ and such that the union of all $B_{\tilde{f}}$ is closed.

*Proof.* Since $\tilde{f} - f \subseteq W_f$ is a compact convex polytope, there is some $g = g_{\tilde{f}} \in \tilde{f}\backslash\partial\tilde{f}$ such that $m_g =: \min(\|g - h\| \mid h \in \partial\tilde{f}) > 0$ is larger than or equal to $m_{g'}$ for all $g' \in \tilde{f}$.

Moreover, for each $h \in P$ there exists some $\varepsilon_h > 0$ such that $p \in P$ and $0 \neq \|h - p\| < \varepsilon_h$ implies $[h,\ h + (\varepsilon_h/\|h - p\|)(p - h)] \subseteq P$. Now $h \in \partial\tilde{f}$ implies $4 \cdot \|h - g\| \geqslant \varepsilon_h + m_g$ since otherwise $\varepsilon_h + m_g > 4 \cdot \|h - g\| \geqslant 2 \cdot \|h - g\| + 2 \cdot m_g$, i.e., $\varepsilon_h > 2\|h - g\| + m_g$ which implies $h \prec h + 2(g - h) \divideontimes g$. Thus $h + 2(g - h) \in \tilde{f}\backslash\partial\tilde{f}$ and $m_g \geqslant m_{h+2(g-h)}$. So we may find some $h_1 \in \partial\tilde{f}$ with $\|h + 2(g - h) - h_1\| \leqslant m_g$, in particular, $\|h - h_1\| \leqslant 2\|g - h\| + m_g < \varepsilon_h$ and thus $h \prec h_1$. But this implies $[h, h_1] \prec h_1$ and so it implies $[h, h_1] \subseteq \partial\tilde{f}$ since $h_1 \in \partial\tilde{f}$ implies $\tilde{h}_1 \subseteq \partial\tilde{f}$. In particular $\frac{1}{2}(h + h_1) \in \partial\tilde{f}$ and thus $m_g \leqslant \|g - \frac{1}{2}(h + h_1)\| = \frac{1}{2}\|h + 2(g - h) - h_1\| \leqslant \frac{1}{2}m_g$, a contradiction.

Now put $\varepsilon_{\tilde{f}} = \min(\varepsilon_g/2,\ m_g/4)$. It follows that $B_{\tilde{f}}$ is contained in $\tilde{f}\backslash\partial\tilde{f} = \tilde{g}\backslash\partial\tilde{g}$ since $\|h - g\| \leqslant \varepsilon_g/2$ implies $g \prec h$ and thus $W_g \subseteq W_h$, so $\|h - g\| \leqslant \varepsilon_g/2$ and $h \in P_n$ implies $g \prec h$ and $W_g = W_h$ and thus $g \divideontimes h$, i.e., $h \in \tilde{f}\backslash\partial\tilde{f}$. Moreover, the union of all the $B_{\tilde{f}}$ is closed, since $h_i \in \bigcup B_{\tilde{f}}$ and $h_i \to h$ ($i \in \mathbb{N}$) implies $h \in P_n$ and $\|h - h_i\| < \varepsilon_h/4$ as well as $h \prec h_i$ for almost all $i$ which together with $g_i = g_{\tilde{h}_i}$ and $\|h_i - g_i\| \leqslant m_{g_i}/4$ implies $h \prec g_i$ and $\|h - g_i\| < (\varepsilon_h + m_{g_i})/4$ and therefore $h \divideontimes g_i$ for almost all $i$ (since otherwise $h \in \partial\tilde{g}_i$ and so $4\|h - g_i\| \geqslant \varepsilon_h + m_{g_i}$). But this implies $g_i = g_{i_0}$ for some fixed $i_0$ and almost all $i$ and thus

$$h \in B_{\tilde{g}_{i_0}} \subseteq \bigcup B_{\tilde{f}}.$$

**(6.8)** It now follows from standard arguments in topology that $\dim P_n = n$ unless $P = P_{n-1}$ and that in case $\dim P < \infty$ one can compute the (co-)homology of $P$ from the following chain complex: For each $n \in \mathbb{N}$ let $S_n = S_n(P)$ denote the set of pairs $(f, o_f)$, where $f$ is an element in $P$ with $\dim W_f = n$ and $o_f$ is an orientation of $W_f$—if $W_f = \{0\}$ just assume $o_f \in \{\pm 1\}$.

Let $C_n = C_n(P)$ denote the free abelian group generated by $S_n$ modulo the subgroup generated by all sums $(f, o_g) + (g, o_g)$, where $f \divideontimes g$ and $o_f$ and $o_g$ are opposite orientations of $W_f = W_g$. Define $\partial\colon C_n \to C_{n-1}$ by $\partial(f, o_f) = \sum_{i=1}^{k} (g_i, o_i)$, where $g_1, \ldots, g_k$ is a system of representatives of the $\divideontimes$-equivalence classes of elements $g \in \partial\tilde{f}$ with $\dim W_g = n - 1$ and $o_i$ is the (unique) orientation of $W_i = W_{g_i}$ for which a basis $v_1, ..., v_{n-1}$ of $W_i$ is positively oriented if and only if the basis $v_1, ..., v_{n-1}, f - g_i$ of $W_f = W_i \oplus \langle f - g_i \rangle$ is positively oriented relative to $o_f$. Then one has $\partial \cdot \partial = 0$ and the (co-)homology of $P$ coincides with the (co-)homology of the complex $C_*(P)$.

Another chain complex which determines the (co-)homology of $P$ can be derived from the barycentric subdivision of the cell decomposition $P = \bigcup_{f \in P} \tilde{f}$: let $F_n = F_n(P)$ denote the set of all sequences $(\tilde{f}_0, \tilde{f}_1, ..., \tilde{f}_n)$ with $f_i \in P$ $(i = 1, ..., n)$ and $\tilde{f}_0 \subsetneq \tilde{f}_1 \subsetneq \cdots \subsetneq \tilde{f}_n$, let $B_n = B_n(P)$ denote the free abelian group, generated by $F_n$, and define $\partial$: $B_n \to B_{n-1}$ by $\partial(\tilde{f}_0, ..., \tilde{f}_n) = \sum_{i=0}^{n} (-1)^i (\tilde{f}_0, ..., \hat{\tilde{f}}_i, ..., \tilde{f}_n)$. Again, one has $\partial \cdot \partial = 0$ and the (co-)homology of $P$ can be identified with the (co-)homology of $B_*(P)$, as well.

**(6.9)** Let us now assume $P = T_X$ for some strongly discrete metric space $X$ of finite combinatorial dimension. Then $C_*(X) =: C_*(T_X)$ and $B_*(X) =: B_*(T_X)$ are exact except in dimension 0 since $T_X$ is contractible. We can reinterpret the chain complexes $C_*(X)$ and $B_*(X)$ in this case: we know that dim $W_f = k$ for some $f \in T_X$ if and only if rk $f = k$ and that $f < g$ for $f, g \in T_X$ if and only if $\mathscr{K}_f \supseteq \mathscr{K}_g$. Thus we can construct $C_*(X)$ and $B_*(X)$ from the partially ordered set $\mathscr{R}_X = \{\mathscr{K}_f \mid f \in T_X\} = \{\mathscr{K} \subseteq X \times X \mid xy = xy_{\mathscr{K}}$ for all $(x, y) \in \mathscr{K}$ and $xy > xy_{\mathscr{K}}$ for all $(x, y) \in (X \times X) \backslash \mathscr{K}\} \subseteq \mathscr{P}(X \times X)$ of "admissible relations" on $X$ which is considered to be partially ordered by inverse inclusion (i.e., $\mathscr{K} \leqslant \mathscr{L} \Leftrightarrow \mathscr{K} \supseteq \mathscr{L}$) in the following way: let $S_k = S_k(X)$ denote the set of pairs $(\mathscr{K}, o_{\mathscr{K}})$ with $\mathscr{K} \in \mathscr{R}_X$, rk $\mathscr{K} = k$ and with $o_{\mathscr{K}} = (x_1, ..., x_k)$ denoting a sequence of $k$ elements in $X$, one out of each bipartite connected component (and $o_{\mathscr{K}} \in \{\pm 1\}$ if $k = 0$). For any two such sequences $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$ let sgn$((x_1, ..., x_k), (y_1, ..., y_k))$ denote the product of the signum of the permutation $\pi \in \Sigma_k$ for which $(x_i, y_{\pi(i)}) \in \mathscr{K}^{n_i}$ for some $n_i \in \mathbb{N}$ with $\prod_{i=1}^{k} (-1)^{n_i}$. Note that each such sequence $o_{\mathscr{K}} = (x_1, ..., x_k)$ defines an orientation $o_{W_{\mathscr{K}}}$ of $W_{\mathscr{K}}$ for which a basis $v_1, ..., v_k \in W_{\mathscr{K}}$ is positively oriented if and only if the determinant $\det(v_i(x_j))_{i,j=1,...,k}$ is positive and that sgn$(o_{\mathscr{K}}, o'_{\mathscr{K}}) = +1$ if and only if $o_{W_{\mathscr{K}}} = o'_{W_{\mathscr{K}}}$.

Now $C_k(X)$ can be identified with the free abelian group, generated by $S_k(X)$, modulo the subgroup, generated by all expressions of the form $(\mathscr{K}, o_{\mathscr{K}}) + (\mathscr{K}, o'_{\mathscr{K}})$ with sgn$(o_{\mathscr{K}}, o'_{\mathscr{K}}) = -1$ in which case $\partial$: $C_k(X) \to C_{k-1}(X)$ maps some generator $(\mathscr{L}, o_{\mathscr{L}} = (x_1, ..., x_k)) \in S_k(X)$ onto the sum $\sum_{\mathscr{K} \in \mathscr{R}_X, \mathscr{L} \subseteq \mathscr{K}, \mathrm{rk}\, \mathscr{K} = k-1} (-1)^{i_{\mathscr{K}}} \cdot \mathrm{sgn}(v_{\mathscr{K}}(x_{i_{\mathscr{K}}})) \cdot (\mathscr{K}, (x_1, ..., \hat{x}_{i_{\mathscr{K}}}, ..., x_k))$, where $i_{\mathscr{K}} \in \{1, ..., k\}$ is some index such that $x_1, ..., \hat{x}_{i_{\mathscr{K}}}, ..., x_k$ is a system of representatives of the $(k-1)$ bipartite connected components of $\mathscr{K}$ and $v_{\mathscr{K}} \in V_{\mathscr{L}}^{\mathscr{K}} \backslash V_{\mathscr{K}}$.

Similarly, $B_k(X)$ can be identified with the free abelian group, generated by all sequences $(\mathscr{K}_0, \mathscr{K}_1, ..., \mathscr{K}_k) \in \mathscr{R}_X^{k+1}$ with $\mathscr{K}_0 \supsetneq \mathscr{K}_1 \supsetneq \cdots \supsetneq \mathscr{K}_k$, in which case $\partial\colon B_k(X)\to B_{k-1}(X)$ maps a generator $(\mathscr{K}_0,\mathscr{K}_1,\ldots,\mathscr{K}_k)$ onto the sum $\sum_{i=0}^{k}(-1)^i(\mathscr{K}_0,\ldots,\hat{\mathscr{K}}_i,\ldots,\mathscr{K}_j)$.

It follows that in case $\dim_{\text{comb}}X=n$ the complexes

$$0\longleftarrow\mathbb{Z}\overset{d}{\longleftarrow}C_0(X)\overset{\partial}{\longleftarrow}C_1(X)\overset{\partial}{\longleftarrow}\cdots\overset{\partial}{\longleftarrow}C_n(X)\longleftarrow 0$$

and

$$0\longleftarrow\mathbb{Z}\overset{d}{\longleftarrow}B_0(X)\overset{\partial}{\longleftarrow}B_1(X)\longleftarrow\cdots\overset{\partial}{\longleftarrow}B_n(X)\longleftarrow 0$$

are exact, if $d\colon C_0(X)\to\mathbb{Z}$ is defined by $d(\mathscr{K},o_{\mathscr{K}})=o_{\mathscr{K}}\in\{\pm 1\}$ if $\mathscr{K}\in\mathscr{R}_X$ and $\operatorname{rk}\mathscr{K}=0$ and $d\colon B_0(X)\to\mathbb{Z}$ by $d((\mathscr{K}_0))=+1$.

Note that the complex $C_*(X)$ can also be described purely in terms of the partial order, defined on $\mathscr{R}_X$ by inverse inclusion: for each $\mathscr{K}\in\mathscr{R}_X^{(k)}=:\{\mathscr{K}\in\mathscr{R}_X\mid\operatorname{rk}\mathscr{K}=k\}$ let $\mathscr{F}_{\mathscr{K}}=:\{(\mathscr{K}_0,\mathscr{K}_1,\ldots,\mathscr{K}_k=\mathscr{K})\in\mathscr{R}_X^{k+1}\mid\mathscr{K}_0\supsetneqq\mathscr{K}_1\supsetneqq\cdots\supsetneqq\mathscr{K}_k=\mathscr{K}\}$ denote the set of all maximal linearly ordered sequences ending with $\mathscr{K}$ and define an orientation $\omega_{\mathscr{K}}$ of $\mathscr{K}$ to be a map $\omega_{\mathscr{K}}\colon\mathscr{F}_{\mathscr{K}}\to\{\pm 1\}$ such that $\omega_{\mathscr{K}}(\mathscr{K}_0,\mathscr{K}_1,\ldots,\mathscr{K}_k=\mathscr{K})=-\,\omega_{\mathscr{K}}(\mathscr{L}_0,\mathscr{L}_1,\ldots,\mathscr{L}_k=\mathscr{K})$ whenever $\#\{i\in\{0,\ldots,k\}\mid\mathscr{K}_i\neq\mathscr{L}_i\}=1$. One can show that there are precisely two orientations for each $\mathscr{K}\in\mathscr{R}_X$, which differ by their sign, only, that any sequence $o_{\mathscr{K}}=(x_1,\ldots,x_{\mathscr{K}})$ defines an orientation $\omega_{x_1,\ldots,x_k}$ of $\mathscr{K}$ and that $\operatorname{sgn}((x_1,\ldots,x_k),(y_1,\ldots,y_k))\cdot\omega_{x_1,\ldots,x_k}=\omega_{y_1,\ldots,y_k}$ whenever $x_1,\ldots,x_k$ and $y_1,\ldots,y_k$ are two different systems of representatives of the $k$ bipartite connected components of $\mathscr{K}$. Thus we can reinterpret $C_k(X)$ again as the free abelian group, generated by all pairs $(\mathscr{K},\omega_{\mathscr{K}})$ with $\mathscr{K}\in\mathscr{R}_X^{(k)}$ and $\omega_{\mathscr{K}}$ an orientation of $\mathscr{K}$, modulo the subgroup, generated by all sums of the form $(\mathscr{K},\omega_{\mathscr{K}})+(\mathscr{K},-\omega_{\mathscr{K}})$, in which case $\partial\colon C_k(X)\to C_{k-1}(X)$ is defined by

$$\partial(\mathscr{K},\omega_{\mathscr{K}})=:\sum_{\mathscr{L}\in\mathscr{R}_X^{(k-1)},\,\mathscr{L}\supseteq\mathscr{K}}(\mathscr{L},\omega_{\mathscr{L}}^{\mathscr{K}}),$$

where $\omega_{\mathscr{L}}^{\mathscr{K}}\colon\mathscr{F}_{\mathscr{L}}\to\{\pm 1\}$ is defined by $\omega_{\mathscr{L}}^{\mathscr{K}}(\mathscr{L}_0,\mathscr{L}_1,\ldots,\mathscr{L}_{k-1}=\mathscr{L})=:\omega_{\mathscr{K}}(\mathscr{L}_0,\mathscr{L}_1,\ldots,\mathscr{L}_{k-1},\mathscr{K})$.

It is now rather easy to derive Theorem 10 from these considerations. One just has to remark that the group $G$ acts freely on $\mathscr{R}_{(G,l)}=\{\mathscr{K}_f\subseteq G\times G\mid f\in T_{(G,l)}\}$. But if $f\in T_{(G,l)}$, $g\in G$, $g\mathscr{K}_f=\mathscr{K}_f$, and, say, $(1,x)\in\mathscr{K}_f$, it follows that $(g^n,g^nx)\in\mathscr{K}_f$ and thus $f(g^n)+f(g^nx)=l((g^n)^{-1}g^nx)=l(x)$, which in view of $l(g^n)\leqslant f(1)+f(g^n)$ implies $l(g^n)\leqslant 2l(x)$ for all $n\in\mathbb{Z}$ and thus $\#\{g^n\mid n\in\mathbb{Z}\}<\infty$, i.e., $g=1$, since $G$ was supposed to the torsion free.

*Remark.* It may be interesting to study the action of $G$ on the contractible cell complex $T_{(G,l)}$ even for finite groups $G$ and in this way to relate properties of length functions $l\colon G\to\mathbb{N}$, defined on $G$, with other properties of $G$.

# APPENDIX: ON THE EXISTENCE AND FINITENESS OF OPTIMAL NETWORKS, REALIZING A FINITE METRIC SPACE

The existence and finiteness of optimal networks (or "weighted graphs") realizing a finite metric space is being proved and some examples are discussed, including a simple counterexample to a conjecture of Hakimi and Yau.

## A1. INTRODUCTION

Let $X$ be a metric space with distance map $D: X \times X \to \mathbb{R}: (x, y) \mapsto xy$. We want to study "realizations" of $X$ by "networks" (cf. [9, 10, 13, 14, 28, 31], i.e., by systems $\mathcal{N} = (V, \mathscr{E}, l)$, consisting of a set $V = V_{\mathcal{N}}$, the vertices of $\mathcal{N}$, a subset $\mathscr{E} = \mathscr{E}_{\mathcal{N}}$ of $\mathscr{P}_2(V) = \{e \subseteq V \mid \#e = 2\}$, the edges of $\mathcal{N}$, and a map $l = l_{\mathcal{N}}: \mathscr{E} \to \mathbb{R}_+ =: \{r \in \mathbb{R} \mid r \geqslant 0\}$.

For any such network $\mathcal{N} = (V, \mathscr{E}, l)$ and any subset $\mathscr{E}' \subseteq \mathscr{E}$ we define the span $\|\mathscr{E}'\|$ of $\mathscr{E}'$ as the sum $\sum_{e \in \mathscr{E}'} l(e) \in \mathbb{R} \cup \{\infty\}$. In case $\mathscr{E}' = \mathscr{E}$ we also write $\|\mathcal{N}\|$ instead of $\|\mathscr{E}\|$.

For a network $\mathcal{N} = (V, \mathscr{E}, l)$ and any two elements $u, v \in V$ let $\mathcal{N}_{u,v} \subseteq \bigcup_{n \geqslant 1} V^n$ denote the set of "nonrepetitive paths" from $u$ to $v$ in $\mathcal{N}$, i.e., the set of finite sequences $(v_1, v_2, \ldots, v_n) \in V^n$ $(n \geqslant 1)$ with $v_1 = u$, $v_n = v$, $\{v_{\nu-1}, v_\nu\} \in \mathscr{E}$, and $v_\mu \neq v_\nu$ for all $1 \leqslant \mu < \nu \leqslant n$. Obviously, a nonrepetitive path $\not{p} = (v_1, \ldots, v_n) \in \mathcal{N}_{u,v}$ is uniquely determined by its set of edges $\{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\} \subseteq \mathscr{E}$ which will thus also be denoted by $\not{p}$. In the following all paths $\not{p}$ to be considered will be assumed to be nonrepetitive.

A network $\mathcal{N} = (V, \mathscr{E}, l)$ is said to be proper, if $l(e) \neq 0$ for all $e \in \mathscr{E}$ and if $\deg_{\mathcal{N}} v =: \#\{e \in \mathscr{E} \mid v \in e\}$ is at least 1 for each $v \in V$, i.e., if $V = \bigcup_{e \in \mathscr{E}} e$.

For any $u, v \in V$ we define

$$\begin{aligned} \overline{uv} &= 0 && \text{if} \quad u = v, \\ &= \inf(\|\not{p}\| \mid \not{p} \in \mathcal{N}_{u,v}) && \text{if} \quad u \neq v \text{ and } \mathcal{N}_{u,v} \neq \varnothing, \\ &= \infty && \text{if} \quad u \neq v \text{ and } \mathcal{N}_{u,v} = \varnothing. \end{aligned}$$

A path $\not{p} \in \mathcal{N}_{u,v}$ is said to be a geodesic if $\|\not{p}\| = \overline{uv}$ and $l(\{w, w'\}) \neq 0$ for all $\{w, w'\} \in \not{p}$. Let $\bar{\mathcal{N}}_{u,v}$ denote the set of geodesics in $\mathcal{N}_{u,v}$.

A network $\mathcal{N} = (V, \mathscr{E}, l)$ is said to realize the metric space $X$ if $X$ is contained in $V$ and one has $xy = \overline{xy}$ for all $x, y \in X$.

Let $\mathcal{N}(X)$ denote the class of networks realizing $X$ and define the span $\|X\|$ of $X$ as the infimum $\|X\| =: \inf(\|\mathcal{N}\| \mid \mathcal{N} \in \mathcal{N}(X))$. A network $\mathcal{N} \in \mathcal{N}(X)$ is said to be an optimal realization of $X$ if $\|\mathcal{N}\|$ coincides with $\|X\|$.

In this note we want to prove the following, intuitively almost obvious, but still seemingly not quite trivial

THEOREM. *If $X$ is a finite metric space, then there exist optimal proper networks $\mathcal{N}$ realizing $X$, any such network $\mathcal{N}=(V,\mathcal{E},l)$ is finite (i.e., the set $V$ is finite) and $V$ contains at most $N(N-1)^2(N-2)/4$ vertices $v$ with $2<\deg_{\mathcal{N}}v$, if $\#X=N$. Moreover, there are finitely many optimal proper networks $\mathcal{N}_1=(V_1,\mathcal{E}_1,l_1),\ldots,\mathcal{N}_k=(V_k,\mathcal{E}_k,l_k)\in\mathcal{N}(X)$ such that for any optimal proper network $\mathcal{N}=(V,\mathcal{E},l)\in\mathcal{N}(X)$ with $\deg_{\mathcal{N}}v\neq 2$ for all $v\in V\backslash X$ there is some $\kappa\in\{1,\ldots,k\}$ and some bijection $\psi\colon V\simeq V_\kappa$ with $\psi(x)=x$ for all $x\in X$ such that $(v_1,v_2,\ldots,v_n)$ is a geodesic in $\mathcal{N}$ if and only if $(\psi(v_1),\psi(v_2),\ldots,\psi(v_n))$ is a geodesic in $\mathcal{N}_\kappa$.*

*Remark* 1. For any such $\psi\colon V\mapsto V_\kappa$ one has obviously $\{\psi(u),\psi(v)\}\in\mathcal{E}_\kappa$ if and only if $\{u,v\}\in\mathcal{E}$ for all $u,v\in V$. I conjecture that one has also $l_\kappa(\{\psi(u),\psi(v)\})=l(\{u,v\})$ for any $\{u,v\}\in\mathcal{E}$. Moreover I conjecture that for any $n$ there is an open and dense subset $\mathcal{O}$ in $\{D\colon\{1,\ldots,n\}^2\to\mathbb{R}\mid D$ a metric$\}\subseteq\mathbb{R}^{n^2}$ such that for all metric spaces $X=(\{1,\ldots,n\},D)$ with $D\in\mathcal{O}$ there is—up to isomorphism—only one proper optimal realization $\mathcal{N}=(V,\mathcal{E},l)\in\mathcal{N}(X)$ with $\deg v\neq 2$ for all $v\in V$ (cf. [10]). That there are also finite metric spaces which have more than one minimal realization is indicated in Section 3, thereby settling a conjecture of Hakimi and Yau (cf. [10]) in the negative.

*Remark* 2. The analysis presented in Section 2 allows in principle the construction of the networks $\mathcal{N}_1,\ldots,\mathcal{N}_k$ for any given finite metric space $X$ in finitely many steps. It seems worthwhile to ask for more efficient algorithms and to discuss the complexity of this problem.

## A2. PROOF OF THE THEOREM

In this section let $X$ be a fixed finite metric space. We start with the following trivial observation:

**(A2.1)** If $\mathcal{N}=(V,\mathcal{E},l)\in\mathcal{N}(X)$, then one has $\bar{\mathcal{N}}_{x,y}\neq\varnothing$ for all $x,y\in X$ with $x\neq y$ if and only if there exist finite subsets $V_0\subseteq V$ and $\mathcal{E}_0\subseteq\mathcal{E}\cap\mathcal{P}_2(V_0)$ such that the finite "subnetwork" $\mathcal{N}_0=(V_0,\mathcal{E}_0,l_0=l|_{\mathcal{E}_0})$ is proper and still in $\mathcal{N}(X)$. Moreover, in this case one can choose $V_0$ and $\mathcal{E}_0$ in such a way, that $\mathcal{N}_0$ is "tight," i.e., in such a way that $(V_0,\mathcal{E}_1=\mathcal{E}_0\backslash\{e\}, l_1=l|_{\mathcal{E}_1})$ is not in $\mathcal{N}(X)$ for all $e\in\mathcal{E}_0$.

*Proof.* If such a proper finite subnetwork exists, one has obviously $\bar{\mathcal{N}}_{x,y}\neq\varnothing$ for all $x,y\in X$ with $x\neq y$. Vice versa, if $\bar{\mathcal{N}}_{x,y}\neq\varnothing$ for all $x,y\in X$

with $x \neq y$, one just chooses a geodesic $\not{p}_{\{x,y\}}$ for each $\{x, y\} \in \mathscr{P}_2(X)$ and puts $\mathscr{E}_0 = \bigcup_{\{x,y\} \in \mathscr{P}_2(X)} \not{p}_{\{x,y\}}$ and $V_0 = \bigcup_{e \in \mathscr{E}_0} e$. Finally, if the resulting network $\mathscr{N}_0$ is not tight, one just chooses a minimal subset $\mathscr{E}' \subseteq \mathscr{E}_0$ such that $\mathscr{N}' = (\bigcup_{e \in \mathscr{E}'} e,\ \mathscr{E}',\ l' = l|_{\mathscr{E}'})$ is in $\mathscr{N}(X)$ which exists because of the finiteness of $\mathscr{E}_0$. The minimality of $\mathscr{E}'$ then guarantees the tightness of $\mathscr{N}'$.

Another trivial, but useful observation is

**(A2.2)** If $\not{p} = (v_1, ..., v_n)$ is geodesic in some network $\mathscr{N}$ and if $1 \leqslant \nu < \mu \leqslant n$, then $(v_\nu, v_{\nu+1}, ..., v_\mu)$ is a geodesic, too. In particular, if $\not{p} = (v_1, ..., v_n)$ and $\not{q} = (w_1, ..., w_m)$ are geodesics in $\mathscr{N}$ and if $v_\nu = w_\alpha$ and $v_\mu = w_\beta$ for some $\nu, \mu \in \{1, ..., n\}$ and $\alpha, \beta \in \{1, ..., m\}$ with $\nu < \mu$ and $\alpha < \beta$ then $(w_1, ..., w_\alpha = v_\nu,\ v_{\nu+1}, ..., v_\mu = w_\beta,\ w_{\beta+1}, ..., w_m)$ is a geodesic, too, and hence, by choosing $\nu$ and $\alpha$ as small as possible and $\mu$ and $\beta$ as big as possible, we can always find a geodesic $\not{q}' \in \bar{\mathscr{N}}_{w_1, w_m}$ with $\#\{e \in \not{q}' \backslash \not{p} \mid e \cap \bigcup_{f \in \not{p}} f \neq \varnothing\} \leqslant 2$.

Next we show

**(A2.3)** If $\mathscr{N} \in \mathscr{N}(X)$ is proper and tight, then it is finite; in particular, any optimal realization $\mathscr{N}$ of $X$ is finite.

*Proof.* By (A2.1), it is enough to show that $\bar{\mathscr{N}}_{x,y} \neq \varnothing$ for all $x, y \in X$ with $x \neq y$. To this end we define for any $u, v \in V$ with $\{u, v\} \in \mathscr{E}$ and any $\varepsilon > 0$ the set

$$\mathscr{K}^u_v(\varepsilon) = \{(x, y) \in X^2 \mid \text{there is a path } \not{p} = (v_1 = x, v_2, ..., v_n = y) \in \mathscr{N}_{x,y},$$

with $\|\not{p}\| < xy + \varepsilon$ and $u = v_\nu$, $v = v_{\nu+1}$ for some $\nu \in \{1, ..., n-1\}$.

Since $\mathscr{N}$ is tight, we have $\mathscr{K}^u_v(\varepsilon) \neq \varnothing$ for all $u, v \in V$ with $\{u, v\} \in \mathscr{E}$ and any $\varepsilon > 0$. Since $\mathscr{K}^u_v(\varepsilon')$ is contained in $\mathscr{K}^u_v(\varepsilon)$ for $\varepsilon' < \varepsilon$ and since $\mathscr{K}^u_v(\varepsilon)$ is finite together with $X$, we have that even the intersection $\mathscr{K}^u_v = \bigcap_{\varepsilon > 0} \mathscr{K}^u_v(\varepsilon)$ is not empty for any $u, v \in V$ with $\{u, v\} \in \mathscr{E}$.

Note that $(x, y) \in \mathscr{K}^u_v(\varepsilon)$ implies $xy \leqslant \overline{xu} + \overline{uv} + \overline{vy} \leqslant \overline{xu} + l(\{u, v\}) + \overline{vy} \leqslant xy + \varepsilon$ and this in turn implies easily

$$\mathscr{K}^u_v = \{(x, y) \in X^2 \mid xy = \overline{xu} + l(\{u, v\}) + \overline{vy}\}.$$

In particular, $\mathscr{K}^u_v \neq \varnothing$ implies $l(\{u, v\}) = \overline{uv}$.

Next we claim $\deg(v) = \#\{e \in \mathscr{E} \mid v \in e\} < \infty$ for each $v \in V$. Otherwise there would exist at least one $v \in V$ and two elements $u_1, u_2 \in V$ with $\{v, u_1\}, \{v, u_2\} \in \mathscr{E}$, and $\mathscr{K}^v_{u_1} = \mathscr{K}^v_{u_2}$, in which case the edge $\{v, u_2\}$ is superfluous in $\mathscr{N}$—contradicting the tightness of $\mathscr{N}$—since it is surely superfluous for any $(x, y) \in X^2 \backslash \mathscr{K}^v_{u_1}$, whereas for the remaining pairs $(x, y) \in \mathscr{K}^v_{u_1} = \mathscr{K}^v_{u_2}$ there exists for any $\varepsilon > 0$ a path $\not{p} \in \mathscr{N}_{x,y}$ with $\|\not{p}\| < xy + \varepsilon$ and with $\{v, u_1\} \in \not{p}$ which implies $\{v, u_2\} \notin \not{p}$, at least for $\varepsilon < \mathrm{Min}(\overline{vu_1}, \overline{vu_2})$.

Next we define a pair $(u, v) \in V^2$ with $\{u, v\} \in \mathscr{E}$ to be unavoidable with respect to some $(x, y) \in \mathscr{K}_v^u$ if there exists some $\varepsilon > 0$ such that for any path $\not{h} \in \mathscr{N}_{x,y}$ with $\|\not{h}\| < xy + \varepsilon$ one has necessarily $\{u, v\} \in \not{h}$. Let $\bar{\mathscr{K}}_v^u$ denote the set of those $(x, y) \in \mathscr{K}_v^u$ for which $(u, v)$ is unavoidable. Since $\mathscr{N}$ is supposed to be tight we have have $\bar{\mathscr{K}}_v^u \neq \varnothing$ for all $u, v \in V$ with $\{u, v\} \in \mathscr{E}$. If $(x, y) \in \bar{\mathscr{K}}_{v_1}^{u_1} \cap \bar{\mathscr{K}}_{u_2}^{u_2}$ for some $u_1, v_1, u_2, v_2 \in V$ with $\{u_1, v_1\}$, $\{u_2, v_2\} \in \mathscr{E}$, then one has necessarily either $xy = \overline{xu_1} + \overline{u_1 v_1} + \overline{v_1 u_2} + \overline{u_2 v_2} + \overline{v_2 y}$ or $xy = \overline{xu_2} + \overline{u_2 v_2} + \overline{v_2 u_1} + \overline{u_1 v_1} + \overline{v_1 y}$.

Now assume $\bar{\mathscr{N}}_{x,y} = \varnothing$ for some $x, y \in X$ with $x \neq y$. For any $v \in V$ define

$$\varepsilon_v = \min(1, l(\{v, w\}) + \overline{wy} - \overline{vy} \mid \{v, w\} \in \mathscr{E}, l(\{v, w\}) + \overline{wy} - \overline{vy} > 0).$$

Thus $\varepsilon_v > 0$ since $\deg v < \infty$ and for any path $\not{h} = (v = v_1, v_2,..., v_n = y)$ in $\mathscr{N}_{v,y}$ with $\|\not{h}\| < \overline{vy} + \varepsilon_v$ one has necessarily $\overline{vv_2} + \overline{v_2 y} = \overline{vy}$.

Now choose a path $\not{h} = (x = v_1, v_2,..., v_n = y) \in \mathscr{N}_{x,y}$ with $\|\not{h}\| < xy + \varepsilon_x$. Since $\overline{xv_2} + \overline{v_2 y} = xy < \overline{v_1 v_2} + \overline{v_2 v_3} + \cdots + \overline{v_{n-1} v_n}$ there is a largest $\nu \in \{2,..., n-1\}$ with $xy = \overline{xv_2} + \overline{v_2 v_3} + \cdots + \overline{v_{\nu-1} v_\nu} + \overline{v_\nu y}$ and this $\nu$ is necessarily smaller than $n - 1$, so one has $xy = \overline{xv_\nu} + \overline{v_\nu y} < \overline{xv_\nu} + \overline{v_\nu v_{\nu+1}} + \overline{v_{\nu-1} y}$ and, thus, $\overline{v_\nu v_{\nu+1}} + \overline{v_{\nu+1} y} \geqslant \overline{v_\nu y} + \varepsilon_{v_\nu}$. Put $w_1 = v_\nu$ and $u_1 = v_{\nu+1}$, forget the old path $\not{h} = (v_1,..., v_n)$ and choose a new path $\not{h} = (w_1 = v_1, v_2,..., v_n = y)$ with $\|\not{h}\| < w_1 y + \varepsilon_{w_1}$. Again there exist some $\nu \in \{2,..., n-2\}$ with $\overline{w_1 y} = \overline{v_0 v_1} + \cdots + \overline{v_{\nu-1} v_\nu} + \overline{v_\nu y}$ and $\overline{v_\nu v_{\nu+1}} + \overline{v_{\nu+1} y} \geqslant \overline{v_\nu y} + \varepsilon_{v_\nu}$. Put $w_2 = v_\nu$ and $u_2 = v_{\nu+1}$. Continuing this way we get two infinite sequences

$$w_1, w_2, w_3,..., \qquad \text{and} \qquad u_1, u_2, u_3,...,$$

such that

(i) $xy = \overline{xw_1} + \overline{w_1 w_2} + \cdots + \overline{w_{n-1} w_n} + \overline{w_n y}$ for all $n = 1, 2,...$;

(ii) $\{w_n, u_n\} \in \mathscr{E}$ and $\overline{w_n u_n} + \overline{u_n y} \geqslant \overline{w_n y} + \varepsilon_{w_n}$ for all $n = 1, 2,...$;

(iii) for all $n = 1, 2,...$, there exists a geodesic $\not{h} \in \bar{\mathscr{N}}_{w_{n-1}, w_n}$ (with $w_0 = x$) which does not contain $\{w_n, u_n\}$.

In particular, for all $m, n \in \mathbb{N}$ with $m < n$ one has $\overline{w_m y} = \overline{w_m w_n} + \overline{w_n y} > \overline{w_n y}$ and thus one has $w_n \neq w_m$ as well as $w_n \neq i_m$. It follows that there exist some $m, n \in \mathscr{N}$ with $m < n$ and $\bar{\mathscr{K}}_{u_m}^{w_m} = \bar{\mathscr{K}}_{u_n}^{w_n} =: \mathscr{K}$. We claim that for all $(a, b) \in \mathscr{K}$ we necessarily have $ab = \overline{aw_n} + \overline{w_n u_n} + \overline{u_n w_m} + \overline{w_m u_m} + \overline{u_m b}$ since otherwise we have $ab = \overline{aw_m} + \overline{w_m u_m} + \overline{u_m w_n} + \overline{w_n u_n} + \overline{u_n b}$ and thus $\overline{w_m u_m} + \overline{u_m w_n} = \overline{w_m w_n}$ which implies $xy = \overline{xw_m} + \overline{w_m w_n} + \overline{w_n y} = \overline{xw_m} + \overline{w_m u_m} + \overline{u_m w_n} + \overline{w_n y}$ and hence $\overline{w_m u_m} + \overline{u_m y} = \overline{w_m y}$, a contradiction.

Thus, for all $(a, b) \in \mathscr{K}$ we have indeed $ab = \overline{aw_n} + \overline{w_n u_n} + \overline{u_n w_m} + \overline{w_m u_m} + \overline{u_m b}$ which implies $\overline{w_n u_n} + \overline{u_n w_m} = \overline{w_n w_m}$. But from the property (iii) just above we derive the existence of a geodesic $\not{h} \in \bar{\mathscr{N}}_{w_n, w_m}$ with $\{w_n, u_n\} \notin \not{h}$.

Thus, for any $(a, b) \in \mathscr{K}$ and any path $\mathscr{q} = (a = z_1, z_2, ..., z_h = b)$ in $\mathscr{N}_{a,b}$ with, say, $z_\nu = w_n$, $z_{\nu+1} = u_n$, $z_\mu = w_m$, and $z_{\mu+1} = u_m$ for some $0 \leqslant \nu < \mu \leqslant h$ the part $(z_\nu, z_{\nu+1}, ..., z_\mu)$ in $\mathscr{q}$ can be replaced by the geodesic $\mathscr{p} \in \bar{\mathscr{N}}_{w_m, w_n}$ with $\{w_n, u_n\} \notin \mathscr{p}$ by which replacement we get a path $\mathscr{q}' \in \mathscr{N}_{a,b}$ of length $\|\mathscr{q}'\|$ smaller than or equal to $\|\mathscr{q}\|$ which avoids $\{w_n, u_n\}$, a final contradiction.

Now we can prove

**(A2.4)** For any $\mathscr{N} \in \mathscr{N}(X)$ there is some proper, finite, and tight $\mathscr{N}' \in \mathscr{N}(X)$ with $\|\mathscr{N}'\| \leqslant \|\mathscr{N}\|$ and hence, one has

$$\|X\| = \inf(\|\mathscr{N}\| \mid \mathscr{N} \in \mathscr{N}(X), \mathscr{N} \text{ finite and tight}).$$

*Proof.* If $\mathscr{N}$ is finite, there is nothing to prove. Otherwise we have $\|\mathscr{N}\| > \|X\|$ by (A2.3) and thus we can find some $\mathscr{N}' = (V, \mathscr{E}, l) \in \mathscr{N}(X)$ with $\|\mathscr{N}'\| < \|\mathscr{N}\|$. W.l.o.g. we may assume $\overline{uv} = l(\{u, v\})$ for all $\{u, v\} \in \mathscr{E}$ and $\overline{uv} \neq 0$ for all $u, v \in V$ with $u \neq v$. Again, if $\mathscr{N}'$ is finite, there is nothing left to prove. Otherwise choose some $\varepsilon > 0$ with $\|\mathscr{N}'\| + \varepsilon \cdot \binom{\#X}{2} < \|\mathscr{N}\|$ and some finite subset $\mathscr{E}_0 \subseteq \mathscr{E}$ with $\|\mathscr{E}_0\| > \|\mathscr{N}'\| - \varepsilon$ and $l(e) > 0$ for all $e \in \mathscr{E}_0$. Let $V_0 = X \cup \bigcup_{e \in \mathscr{E}_0} e$. Since $X$ and $V_0$ are finite, the set of nonnegative real numbers $\{\overline{xv_1} + \overline{v_1 v_2} + \cdots + \overline{v_n y} - xy \mid x, y \in X;\ n \in \mathbb{N};\ v_1, ..., v_n \in V_0\}$ is obviously discrete. In particular, there is some positive $\eta$ such that

$$\overline{xv_1} + \overline{v_1 v_2} + \cdots + \overline{v_n y} - xy < \eta$$

for some $x, y \in X$, $n \in \mathbb{N}$, and $v_1, ..., v_n \in V_0$ implies $\overline{xv_1} + \overline{v_1 v_2} + \cdots + \overline{v_n y} - xy = 0$. Now choose for any $x, y \in X$ with $x \neq y$ some path $\mathscr{p} = x = u_1, u_2, ..., u_n = y)$ in $\mathscr{N}'$ with $\|\mathscr{p}\| < xy + \eta$. Let $\mathscr{p} \cap \mathscr{E}_0 = \{\{u_{i_1}, u_{i_1+1}\}, ..., \{u_{i_k}, u_{i_k+1}\}\}$ with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n-1$. From $\|\mathscr{p}\| < xy + \eta$ we get

$$\overline{xu_{i_1}} + \overline{u_{i_1} u_{i_1+1}} + \overline{u_{i_1+1} u_{i_2}} + \overline{u_{i_2} u_{i_2+1}} + \cdots + \overline{u_{i_k+1} y} - xy < \eta$$

and thus $\overline{xu_{i_1}} + \overline{u_{i_1} u_{i_1+1}} + \cdots + \overline{u_{i_k+1} y} = xy$. By the choice of $\mathscr{E}_0$ we have

$$\overline{xu_{i_1}} + \overline{u_{i_1+1} u_{i_2}} + \cdots + \overline{u_{i_k+1} y} < \varepsilon.$$

Thus, if we enlarge $\mathscr{E}_0$ by the edges $\{\{x, u_{i_1}\}, \{u_{i_1+1}, u_{i_2}\}, ..., \{u_{i_k+1}, y\}\} \cap \mathscr{P}_2(V_0)$ and define $l(\{u, v\}) = \overline{uv}$ for any such edge $\{u, v\}$, we enlarge $\|\mathscr{E}_0\|$ by less than $\varepsilon$ and get a geodesic from $x$ to $y$ in the enlarged network. Since this can be done for any $\{x, y\} \in \mathscr{P}_2(X)$, we see that we can enlarge $\mathscr{E}_0$ to some set $\mathscr{E}_1 \subseteq \mathscr{P}_2(V_0)$ and define some $l_1 \colon \mathscr{E}_1 \to \mathbb{R}_+$ in such a way that $\mathscr{N}_1 = (V_0, \mathscr{E}_1, l_1)$ is proper and in $\mathscr{N}(X)$ and satisfies $\|\mathscr{N}_1\| \leqslant \|\mathscr{N}'\| + \binom{\#X}{2} \cdot \varepsilon < \|\mathscr{N}\|$. Finally, w.l.o.g. we may assume $\mathscr{N}_1$ to be tight.

Next we prove

**(A2.5)** If $\mathcal{N} = (V, \mathscr{E}, l) \in \mathcal{N}(X)$ is proper and tight and if $\not{h} = (x = v_1, v_2, \ldots, v_n = y)$ is a geodesic in $\mathcal{N}$ for some $x, y \in X$ with $x \neq y$, then

$$\#\{\nu \in \{1, \ldots, n\} \mid \deg v_\nu \geqslant 3\} \leqslant (N-1)(N-2) \qquad \text{if} \quad N = \#X.$$

*Proof.* By (A2.2) we can find for any $a, b \in X$ with $a \neq b$ a geodesic $\not{q}$ with $\#\{e \in \not{q} \backslash \not{h} \mid e \cap \{v_1, \ldots, v_n\} \neq \varnothing\} \leqslant 2$. Moreover, if $a \in \{x, y\}$ (and $b \notin \{x, y\}$) the same argument yields a geodesic $\not{q}$ with

$$\#\{e \in \not{q} \backslash \not{h} \mid e \cap \{v_1, \ldots, v_n\} \neq \varnothing\} = 1.$$

Moreover, once we have chosen a geodesic $\not{q}_{\{a,b\}}$ for all $\{a, b\} \in \mathscr{P}_2(X)$, it follows from the tightness of $\mathcal{N}$ that $\mathscr{E} = \bigcup_{\{a,b\} \in \mathscr{P}_2(X)} \not{q}_{\{a,b\}}$. Thus, using the especially chosen geodesics $\not{q}$ from above we get that

$$\begin{aligned}
&\#\{e \in \mathscr{E} \backslash \not{h} \mid e \cap \{v_1, \ldots, v_n\} \neq \varnothing\} \\
&\leqslant \sum_{a \in X \backslash \{x,y\}} \#\{e \in \not{q}_{\{a,x\}} \cup \not{q}_{\{a,y\}} \backslash \not{h} \mid e \cap \{v_1, \ldots, v_n\} \neq \varnothing\} \\
&\quad + \sum_{\{a,b\} \in \mathscr{P}_2(X \backslash \{x,y\})} \#\{e \in \not{q}_{\{a,b\}} \backslash \not{h} \mid e \cap \{v_1, \ldots, v_n\} \neq \varnothing\} \\
&\leqslant 2(N-2) + 2 \binom{N-2}{2} = (N-1)(N-2).
\end{aligned}$$

From this, (A2.5) follows immediately.

(A2.5) in turn implies:

**(A2.6)** If $\mathcal{N} = (V. \mathscr{E}, l) \in \mathcal{N}(X)$ is proper and tight and if $\#X = N$, then

$$\#\{v \in V \mid \deg v > 2\} \leqslant \frac{N(N-1)^2(N-1)}{4}.$$

*Proof.* Again we choose a geodesic $\not{q}_{\{x,y\}} \in \bar{\mathcal{N}}_{x,y}$ for each $\{x, y\} \in \mathscr{P}_2(X)$. Since $\mathcal{N}$ is tight we have $\mathscr{E} = \bigcup_{\{x,y\} \in \mathscr{P}_2(X)} \not{q}_{\{x,y\}}$. Since any $v \in V$ with $\deg v > 2$ occurs in at least two of our geodesics and since in each of these geodesics there occur at most $(N-1) \cdot (N-2)$ such $v$, we get that indeed

$$\#\{v \in V \mid \deg v > 2\} \leqslant \frac{1}{2} \binom{N}{2} \cdot (N-1)(N-2) = \frac{N(N-1)^2(N-2)}{4}.$$

We are now ready to prove

**(A2.7)** There exist some $\mathcal{N} \in \mathcal{N}(X)$ with $\|\mathcal{N}\| = \|X\|$.

*Proof.* Otherwise there exists an infinite sequence $\mathcal{N}_i = (V_i, \mathcal{E}_i, l_i)$ $(i \in \mathbb{N})$ of finite, proper, and tight networks in $\mathcal{N}(X)$ with $\|\mathcal{N}_i\| > \|X\|$ and $\lim_{i\to\infty} \|\mathcal{N}_i\| = \|X\|$. W.l.o.g. we may assume that for each $v \in V_i \backslash X$ one has $\deg_{\mathcal{N}_i} v > 2$: if degree $v \leqslant 1$ and $v \notin X$, then $v$ cannot occur in any geodesic connecting some $x, y \in X$ and thus it cannot occur in a proper network at all. If $\deg v = 2$, say $\{e \in \mathcal{E} \mid v \in e\} = \{\{v, u_1\}, \{v, u_2\}\}$, then we can exchange the two edges $\{v, u_1\}$ and $\{v, u_2\}$ for the one edge $\{u_1, u_2\}$ and define $l_i(\{u_1, u_2\}) = l_i(\{u_1, v\}) + l_i(\{v, u_2\})$ and drop $v$ and get again a tight, proper, and finite network in $\mathcal{N}(X)$ of the same span. Thus we have $\# V_i \backslash X \leqslant N(N-1)^2(N-2)/4$ for each $i$.

Now, let $Y$ be some arbitrary set of cardinality $N(N-1)^2(N-2)/4$ which is disjoint from $X$. Note that for each $\mathcal{E} \subseteq \mathcal{P}_2(X \cup Y)$ the set $L_{\mathcal{E}} \subseteq \mathbb{R}^{\mathcal{E}}$ of all $l$: $\mathcal{E} \to [0, \max(xy \mid x, y \in X)]$ with $(X \cup Y, \mathcal{E}, l) \in \mathcal{N}(X)$ is a (possibly empty) compact subset of $\mathbb{R}$, more precisely, $L_{\mathcal{E}}$ is the union of finitely many compact convex subsets $L_{\mathcal{E}}(\mathcal{G})$, where $\mathcal{G}$ runs through all subsets of $\mathcal{P}(\mathcal{E})$ which contain for each $x, y \in X$ with $x \neq y$ some $\not{h} \in \mathcal{G}$ of the form $\not{h} = \{\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}\} \subseteq \mathcal{E}$ and $l \in L_{\mathcal{E}}$ is in $L_{\mathcal{E}}(\mathcal{G})$ whenever any $\not{h} \subseteq \mathcal{E}$ with $\not{h} \in \mathcal{G}$ is a geodesic in $(X \cup Y, \mathcal{E}, l)$. Since the "trace-map" $\mathrm{tr} : L_{\mathcal{E}} \to \mathbb{R}$ : $l \mapsto \sum_{e \in \mathcal{E}} l(e)$ is continuous on $L_{\mathcal{E}}$, there exist some $l_{\mathcal{E}} \in L_{\mathcal{E}}$ with $\mathrm{tr}(l_{\mathcal{E}}) = m_{\mathcal{E}} =: \min(\mathrm{tr}(l) \mid l \in L_{\mathcal{E}})$.

Since we only have finitely many $\mathcal{E} \subseteq \mathcal{P}_2(X \cup Y)$ and since $L_{\mathcal{E}}$ cannot be empty for all such $\mathcal{E}$, there exist some $\mathcal{E} \subseteq \mathcal{P}_2(X \cup Y)$ with $L_{\mathcal{E}} \neq \varnothing$ such that for all $\mathcal{E}' \subseteq \mathcal{P}_2(X \cup Y)$ one has $m_{\mathcal{E}} \leqslant m_{\mathcal{E}'}$. We claim that $\|X\| = m_{\mathcal{E}}$ so that $\mathcal{N} = (X \cup Y, \mathcal{E}, l_{\mathcal{E}})$ is an optimal realization of $X$. Moreover, by dropping all elements $v \in X \cup Y$ with $\deg_{\mathcal{N}} v = 0$ and by identifying all $u, v \in X \cup Y$ for which $\overline{uv}$ equals 0 in $\mathcal{N}$ and by modifying $\mathcal{E}$ and $l_{\mathcal{E}}$ accordingly, we get some optimal network $\mathcal{N}_1 \in \mathcal{N}(X)$ which is also proper.

For the proof of the optimality of $\mathcal{N}$ it is enough to show that $m_{\mathcal{E}} \leqslant \|\mathcal{N}_i\|$ for all $i$. But for any $i$ there exists an injective map $\psi_i$: $V_0 \hookrightarrow X \cup Y$ with $\psi(x) = x$ for all $x \in X$. Let $\mathcal{E}' = \psi(\mathcal{E}) = \{\{\psi(u), \psi(v)\} \mid \{u, v\} \in \mathcal{E}\}$ and let $l'$: $\mathcal{E}' \to \mathbb{R}_+$ be defined by $l'(\{\psi(u), \psi(v)\}) = l_i(\{u, v\})$. Then we have $l' \in L_{\mathcal{E}'}$ and thus we have $m_{\mathcal{E}} \leqslant \mathrm{tr}(l') = \|\mathcal{N}_i\|$. Q.E.D.

Finally, let $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_a\}$ denote the set of subsets $\mathcal{E} \subseteq \mathcal{P}_2(X \cup Y)$ with $m_{\mathcal{E}} = \|X\|$ and $X \subseteq V_{\mathcal{E}} = \bigcup_{e \in \mathcal{E}} e$ and $\#\{e \in \mathcal{E} \mid v \in e\} \neq 2$ for all $v \in V_{\mathcal{E}} \backslash X$. For each $\mathcal{E} = \mathcal{E}_\alpha$ $(\alpha = 1, \ldots, a)$ let $\{\mathcal{G}_1^\alpha, \mathcal{G}_2^\alpha, \ldots, \mathcal{G}_{b_\alpha}^\alpha\}$ denote the set of subsets $\mathcal{G} \subseteq \mathcal{P}(\mathcal{E})$ for which there exist some $l \in L_{\mathcal{E}}$ with $\mathrm{tr}(l) = m_{\mathcal{E}} = \|X\|$ such that $l(e) \neq 0$ for all $e \in \mathcal{E}$—so $\mathcal{N}_l^{\mathcal{E}} =: (V_{\mathcal{E}}, \mathcal{E}, l)$ is proper—and $\mathcal{G}$ is the set of geodesics in $\mathcal{N}_l^{\mathcal{E}}$. For each such $\mathcal{G} = \mathcal{G}_\beta^\alpha$ $(\beta = 1, 2, \ldots, b_\alpha)$ choose some such $l = l_\beta^\alpha \in L_{\mathcal{E}_\alpha}$ and let $\mathcal{N}_\beta^\alpha$ $(\alpha = 1, \ldots, a;\ \beta = 1, \ldots, b_\alpha)$ denote the resulting network $(V_\alpha = V_{\mathscr{E}_\alpha}, \mathscr{E}_\alpha, l^\alpha_\beta)$; of course, we may have $b_\alpha = 0$ for some $\alpha \in \{1,..., a\}$, e.g., for $X = \{a, b, c\}$ and $Y = \{a', b', c'\}$ we have $m_{\mathscr{E}} = \|X\| = \frac{1}{2}(ab + bc + ca)$ for $\mathscr{E} = \{\{a, a'\}, \{b, b'\}, \{c, c'\}, \{a', b'\}, \{b', c'\}, \{c', a'\}\}$, but some $l \in L_{\mathscr{E}}$ satisfies $\operatorname{tr}(l) = m_{\mathscr{E}} = \|X\|$ only if $l(\{a', b'\}) = l(\{b', c'\}) = l(\{c', a'\}) = 0$ (and $l(\{a, a'\}) = \frac{1}{2}(ab + ac - bc)$, $l(\{b, b'\}) = \frac{1}{2}(ab + bc - ac)$, $l(\{c, c'\}) = \frac{1}{2}(ac + bc - ab)$).

Now assume $\mathscr{N} = (V, \mathscr{E}, l) \in \mathscr{N}(X)$ to be a proper optimal realization of $X$ with $\deg v \neq 2$ for all $v \in V \backslash X$ and, as above, choose some injective map $\psi\colon V \hookrightarrow X \cup Y$, with $\psi(x) = x$ for all $x \in X$. Let $\mathscr{E}' = \{\{\psi(u), \psi(v)\} \mid \{u, v\} \in \mathscr{E}\}$ and put $l'\colon \mathscr{E}' \to \mathbb{R}\colon \{\psi(u), \psi(v)\} \mapsto l(\{u, v\})$. It follows that $V' =: \psi(V) = \bigcup_{e' \in \mathscr{E}'} e'$ contains $X$, that $\#\{e' \in \mathscr{E}' \mid v' \in e'\} \neq 2$ for all $v' \in \psi(V) \backslash X$ and that $\operatorname{tr}(l') = \|\mathscr{N}\| = \|X\|$ and hence $m_{\mathscr{E}'} = \|X\|$, so we have $\mathscr{E}' = \mathscr{E}_\alpha$ for some $\alpha \in \{1,..., a\}$. Moreover we have $l'(e') \neq 0$ for all $e' \in \mathscr{E}' = \mathscr{E}_\alpha$, thus there must exist some $\beta \in \{1,..., b_\alpha\}$ with $\mathscr{G}^\alpha_\beta = \{\not{h} \subseteq \mathscr{E}' \mid \not{h}' \text{ a geodesic in } (V', \mathscr{E}', l')\}$.

Hence the networks $\mathscr{N}^\alpha_\beta$ $(\alpha = 1,..., a;\ \beta = 1,..., b_\alpha)$ fulfill the requirements mentioned in the last part of our theorem.

## A3. Some Examples and Counterexamples

**(A3.1)** If $X = \{1, 2, 3\}$, then $\mathscr{N} = (\{0, 1, 2, 3\}, \{\{0, 1\}, \{0, 2\}, \{0, 3\}\}, l)$ with $l(\{0, i\}) = \frac{1}{2}(ij + ik - jk)$ $(\{i, j, k\} = \{1, 2, 3\})$ is an optimal realization of $X$. It is proper if and only if $ij + ik > jk$ for all $i, j, k$ with $\{i, j, k\} = \{1, 2, 3\}$. Otherwise, if, for instance, $12 + 23 = 13$, $\mathscr{N} = (X, \{\{1, 2\}, \{2, 3\}\}, l)$ with $l(\{i, j\}) = ij$ for $i = 2$ and $j = 1$ or $j = 3$ is a proper optimal realization. In both cases the given proper optimal realizations are the only optimal realizations $\mathscr{N} = (V, \mathscr{E}, l)$ with $\deg v \neq 2$ for all $v \in V \backslash X$.

**(A3.2)** If $X = \{a, b, c, d\}$ and if, for instance, $ab + cd \geqslant ac + bd \geqslant ad + bc$ then $\mathscr{N} = \{X \cup \{a', b', c', d'\}, \mathscr{E}, l\}$ with $\mathscr{E} = \{\{a, a'\}, \{b, b'\}, \{c, c'\}\ \{d, d'\}, \{a', c'\}, \{c', b'\}, \{b', d'\}, \{d', a'\}\}$ and $l\colon \mathscr{E} \to \mathbb{R}$ defined by

$$l(\{a', c'\}) = \tfrac{1}{2}(ab + cd - ad - bc) = l(\{d', b'\}),$$
$$l(\{c', b'\}) = \tfrac{1}{2}(ab + cd - ac - bd) = l(\{d', a'\}),$$
$$l(\{a, a'\}) = \tfrac{1}{2}(ac + ad - cd),$$
$$l(\{b, b'\}) = \tfrac{1}{2}(bc + bd - cd),$$
$$l(\{c, c'\}) = \tfrac{1}{2}(ac + bc - ab),$$
$$l(\{d, d'\}) = \tfrac{1}{2}(ad + bd - ab),$$

is known to be an optimal realization of $X$. If it is proper, it is the only optimal realization $\mathscr{N} = (V, \mathscr{E}, l)$ with $\deg(v) \neq 2$ for all $v \in V \backslash X$, up to isomorphism. Otherwise we may have degeneracies, but still we have only one proper optimal realization—up to isomorphism—without vertices of degree 2 except possibly those from $X$.

**(A3.3)** If $X = \{a, b, c, d, e\}$ and if $ab = 7$, $ac = 4$, $ad = 5$, $ae = 3$, $bc = 5$, $bd = 6$, $be = 4$, $cd = 7$, $ce = 3$, $de = 4$, then we have two nonisomorphic proper optimal realizations without vertices $v$ of degree 2 except $v = e$, namely,

$$\mathscr{N}_1 = X \cup \{a', b', c', d', u, v, \mathscr{E}_1, l_1\} \qquad \text{and}$$

$$\mathscr{N}_2 = X \cup \{a', b', c', d', u, v, \mathscr{E}_2, l_2\} \qquad \text{with}$$

$$\mathscr{E}_1 = \{\{a, a'\}, \{b, b'\}, \{c, c'\}, \{d, d'\}, \{a', d'\}, \{b', c'\}, \{a', u\}, \{c', u\}, \{b', v\}, \{d', v\}, \{e, u\}, \{e, v\}\}$$

and

$l_1 : \mathscr{E}_1 \to \mathbb{R}$ defined by $l_1(\{x, y\}) = 1$ for $(x, y) = (a, a'), (c, c'), (a', u), (c', u), (b', v), (d', v), (e, u)$, and $(e, v)$ and

$l_1(\{x, y\}) = 2$ for $(x, y) = (a', d'), (b', c'), (b, b')$, and $(d, d')$

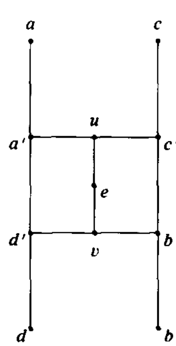
and with

$$\mathscr{E}_2 = \{\{a, a'\}, \{b, b'\}, \{c, c'\}, \{d, d'\}, \{a', c'\}, \{b', d'\}, \{a', u\}, \{d', u\}, \{b', v\}, \{c', v\}, \{e, u\}, \{e, v\}\}$$

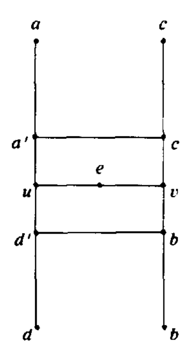
and

$l_2 : \mathscr{E}_2 \to \mathbb{R}$ defined by $l_2(\{x, y\}) = 1$ for $(x, y) = (a, a'), (c, c'), (a', u), (d', u), (b', v), (c', v), (e, u)$, and $(e, v)$ and

$l_2(\{x, y\}) = 2$ for $(x, y) = (a', c'), (b', d'), (b, b')$, and $(d, d')$.




$\mathscr{N}_1$




$\mathscr{N}_2$

Thus, in general, we cannot expect to have only one such minimal realization (cf. [2, Sect. 5]).

**(A3.4)** If $X = \{a, b, c\}$ with $ab = bc = 1$ and $ac = 2$, the family of infinite networks

$$\mathscr{N}_\varepsilon = (\{a = a_0, a_1, a_2, \ldots, b, c = c_0, c_1, c_2, \ldots\}, \mathscr{E}, l_\varepsilon)$$

with

$$\mathscr{E} = \{\{a_{i-1}, a_i\}, \{a_i, c_i\}, \{c_i, b\}, \{c_i, c_{i-1}\} \mid i = 1, 2, \ldots\}$$

and with $l_\varepsilon : \mathscr{E} \to \mathbb{R}$ defined by

$$\begin{aligned} l_\varepsilon(\{a_i, a_{i+1}\}) = l_\varepsilon(\{c_i, c_{i+1}\}) &= 1 - \varepsilon && \text{for} \quad i = 0, \\ &= \varepsilon/2^i && \text{for} \quad i \geqslant 1, \\ l_\varepsilon(\{a_i, c_i\}) = 4 \cdot (\varepsilon/2^i) & && \text{for} \quad i = 1, 2, \ldots, \end{aligned}$$

and

$$l_\varepsilon(\{c_i, b\}) = 2 \cdot (\varepsilon/2^i) \qquad \text{for} \quad i = 1, 2, \ldots,$$

shows that there are infinite proper networks in $\mathscr{N}(X)$ with $\|\mathscr{N}_\varepsilon\| = 2 + 6\varepsilon = \|X\| + 6\varepsilon$ approximating $\|X\|$ arbitrarily well without containing a tight subnetwork and such that any edge in $\mathscr{E}$ occurs in at least one geodesic.

The existence of such networks may explain some of the technical difficulties we encountered in Section 2.

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